ARCHIMEDES MATH COMPETITION

Additional Late Round 2024

Solution:

Problem 1:

The units digit of the entire sum is determined by the units digit of the sum of the units digits of each term. We only need to focus on the last digit of each base number.

Units Digit of 2021^{2021} This is determined by the units digit of $\mathbf{1^{2021}}$. Any integer power of a number ending in 1 will also end in 1.

The units digit is **1**.

Units Digit of 2022^{2022} This is determined by the units digit of $\mathbf{2^{2022}}$. The units digits of powers of 2 follow a cycle of four: (2, 4, 8, 6).

To find where we are in the cycle, we find the remainder of the exponent when divided by 4: 2022 ÷ 4 has a remainder of **2**.

- The second digit in the cycle is 4.
- The units digit is **4**.

Units Digit of 2023^{2023} This is determined by the units digit of $\mathbf{3^{2023}}$. The units digits of powers of 3 follow a cycle of four: (3, 9, 7, 1).

- We find the remainder of the exponent when divided by 4: 2023 ÷ 4 has a remainder of **3**.
- The third digit in the cycle is 7.
- The units digit is **7**.

Units Digit of 2024^{2024} This is determined by the units digit of $\mathbf{4^{2024}}$. The units digits of powers of 4 follow a cycle of two: (4, 6).

- The units digit is 4 for odd exponents and 6 for even exponents.
- Since the exponent **2024** is even, the units digit is **6**.

Sum the Units Digits Finally, add the units digits we found for each term.

- 1 + 4 + 7 + 6 = 18 The units digit of this sum is the final answer.
- The units digit of 18 is **8**. 🔢

Problem 2:

There's a useful mathematical shortcut for this type of problem:

The number of ways a positive integer N can be written as the sum of two or more consecutive positive integers is equal to the number of its odd divisors, excluding 1.

Find the Prime Factorization of 105 First, break down 105 into its prime factors.

- $105 = 3 \times 5 \times 7$

Find the Number of Divisors The prime factorization is $3^1 \times 5^1 \times 7^1$. To find the total number of divisors, we add 1 to each exponent and multiply the results.

- Number of divisors = $(1+1) \times (1+1) \times (1+1) = 2 \times 2 \times 2 = 8$
- Since none of the prime factors are 2, all 8 of these divisors are **odd**.

Apply the Rule The number of ways is equal to the number of odd divisors (8) minus 1 (to exclude the trivial case of the number itself, which is a sum of one term).

- Number of ways = $8 - 1 = 7$

For reference, the 7 ways are:

- 52 + 53
- 34 + 35 + 36
- 19 + 20 + 21 + 22 + 23
- 15 + 16 + 17 + 18 + 19 + 20
- 12 + 13 + 14 + 15 + 16 + 17 + 18
- 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15
- 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 💡

Problem 3:

Find the Largest Number To create the largest possible three-digit number with no repeated digits, we should use the largest digits available (9, 8, 7) and place them in the most significant positions.

- **Hundreds digit:** 9
- **Tens digit:** 8

- **Ones digit:** 7 The largest number is **987**.

Find the Smallest Number To create the smallest possible three-digit number with no repeated digits, we should use the smallest digits available.

- **Hundreds digit:** The first digit cannot be 0, so the smallest possible digit is **1**.
- **Tens digit:** Now we can use the smallest available digit, which is **0**.
- **Ones digit:** The next smallest available digit is **2**. The smallest number is **102**.

Calculate the Difference Finally, subtract the smallest number from the largest number.

- 987 - 102 = 885 🔢

Problem 4:

Calculate the Point Deficit Loretta's target average is 96%, but her average on the first five tests is 91%. This means for each of her first five tests, she is 96 - 91 = 5 points below her goal.

  - Total point deficit = 5 tests × 5 points/test = 25 points. She needs to make up for this 25-point deficit with her future tests.

Calculate the Point Surplus On every upcoming test, Loretta scores 100%. This is 100 - 96 = 4 points *above* her target average. Each new test provides a **4-point surplus** that can be used to cancel out her deficit.

2. **Find the Number of Tests Needed** We need to find the minimum number of tests (n) required for the total surplus to be at least as large as the total deficit.
   - Surplus per test × n ≥ Total Deficit
   - $4 \times n \geq 25$
   - $n \geq 25 / 4$
   - $n \geq 6.25$

Since Loretta must take a whole number of tests, the smallest integer greater than 6.25 is **7**. 📈

Problem 5:

The easiest way to solve this is to find the total number of subsets and then subtract the number of subsets that do **not** meet the condition (the ones that have no prime numbers).

Calculate the Total Number of Subsets The set has **10** integers. The total number of subsets for a set with n elements is 2^n.

- Total subsets = 2^{10} = 1024

Find the Subsets with No Prime Numbers First, let's identify the integers in the set that are **not** prime.

- The prime numbers are {11, 13, 17, 19}.
- The non-prime numbers are **{12, 14, 15, 16, 18, 20}**. There are **6** non-prime numbers. Any subset that contains no primes must be formed using only these 6 numbers. The number of subsets we can form from this group is:
- Subsets with no primes = 2^6 = 64

Find the Final Answer To find the number of subsets with *at least one* prime, we subtract the number of subsets with *no* primes from the total.

- 1024 (total) - 64 (no primes) = 960

Problem 6:

Use the Property of the Golden Ratio The Golden Ratio, let's call it **ϕ**, is defined by the equation $\phi^2 = \phi + 1$. We can use this property to find higher powers of ϕ without having to work with the square root value directly. The n-th power of ϕ can be expressed in terms of Fibonacci numbers (F_n) as $\phi^n = F_n\varphi + F_{n-1}$.

Find the Expression for ϕ^{12} To find ϕ^{12}, we need the 12th and 11th Fibonacci numbers.

- The Fibonacci sequence starts: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, **89 (F_{11})**, **144 (F_{12})**, ...
- Using the formula, $\phi^{12} = F_{12}\varphi + F_{11} = 144\phi + 89$.

Substitute and Simplify Now, substitute the value of the Golden Ratio, $\phi = (1 + \sqrt{5}) / 2$, into our expression.

- $\phi^{12} = 144 * ((1 + \sqrt{5}) / 2) + 89$

- $\phi^{12} = 72 * (1 + \sqrt{5}) + 89$
- $\phi^{12} = 72 + 72\sqrt{5} + 89$
- $\phi^{12} = 161 + 72\sqrt{5}$

Find A + B + C The result is in the form $A + B\sqrt{C}$. By comparing $161 + 72\sqrt{5}$ to this form, we can identify the values:

- **A = 161**
- **B = 72**
- **C = 5** Finally, calculate the sum:
- $A + B + C = 161 + 72 + 5 = 238$ 🧠

Problem 7:

To solve this, we'll count the number of possibilities for each number of digits (1, 2, 3, and 4) and then add them together. The key constraints are that digits **cannot be repeated**, the number must be **divisible by 5** (ending in 0 or 5), and the digits must come from the set **{0, 1, 2, 3, 4, 5, 6}**.

1-Digit Numbers The only positive 1-digit number from the set that is divisible by 5 is **5**.

  - Count = **1**

2-Digit Numbers

  - **Ending in 0:** The last digit is 0. The first digit can be any of the other 6 digits {1, 2, 3, 4, 5, 6}.
    - Count = **6**
  - **Ending in 5:** The last digit is 5. The first digit cannot be 0, so it can be any of the other 5 digits {1, 2, 3, 4, 6}.
    - Count = **5**
  - Total 2-digit numbers = 6 + 5 = 11

3-Digit Numbers

  - **Ending in 0:** The last digit is 0. For the first two positions, we can arrange any 2 of the remaining 6 digits.
    - Choices = $6 \times 5 = 30$
  - **Ending in 5:** The last digit is 5. The first digit cannot be 0, so there are 5 choices for it. The middle digit can be any of the remaining 5 digits.

        - Choices = 5 × 5 = 25
    - Total 3-digit numbers = 30 + 25 = 55

4-Digit Numbers (less than 3000) The first digit must be **1 or 2**.

- **Ending in 0:** The last digit is 0. The first digit can be 1 or 2 (2 choices). After choosing the first and last digit, 5 digits remain for the two middle positions.
    - Choices = 2 × (5 × 4) = 40
- **Ending in 5:** The last digit is 5. The first digit can be 1 or 2 (2 choices). After choosing the first and last digit, 5 digits remain for the two middle positions.
    - Choices = 2 × (5 × 4) = 40
- Total 4-digit numbers = 40 + 40 = 80

Total Count Finally, add the counts from all the cases.

- 1 + 11 + 55 + 80 = 147 🔢

Problem 8:

Set Up the Relationships Let's call the constant sum for each row **R** and the constant sum for each column **S**.

- The sum of all 12 numbers in the grid can be calculated in two ways: by adding the three row sums (3 × R) or by adding the four column sums (4 × S).
- This gives us a key relationship: 3R = 4S.

Find the Connection Between R and S We can find a direct connection between R and S by looking at the row and column containing the letter **A**.

- **Row 1 Sum:** R = 78 + A + 76 + 75 = A + 229
- **Column 2 Sum:** S = A + 81 + 85 = A + 166 By comparing these two equations, we can see that R is 229 - 166 = 63 more than S.
- R = S + 63

Solve for R and S Now we can substitute R = S + 63 into our first relationship, 3R = 4S.

- 3(S + 63) = 4S
- 3S + 189 = 4S

- **S = 189** (This is the sum of each column).
- Now we can find R: R = 189 + 63 = 252 (This is the sum of each row).

Find the Values of A, B, C, D, and E Using the row sum **R=252** and the column sum **S=189**, we can find each missing number.

- **A (from Column 2):** A + 81 + 85 = 189 → A + 166 = 189 → **A = 23**
- **C (from Row 2):** First, we need B. Let's find E first.
- **E (from Column 3):** 76 + 80 + E = 189 → 156 + E = 189 → **E = 33**
- **D (from Row 3):** D + 85 + E + 83 = 252 → D + 85 + 33 + 83 = 252 → D + 201 = 252 → **D = 51**
- **B (from Column 1):** 78 + B + D = 189 → 78 + B + 51 = 189 → B + 129 = 189 → **B = 60**
- **C (from Row 2):** B + 81 + 80 + C = 252 → 60 + 81 + 80 + C = 252 → 221 + C = 252 → **C = 31**

Calculate the Final Sum Finally, add the values of the five letters together.

- A + B + C + D + E = 23 + 60 + 31 + 51 + 33 = 198 🔢

Problem 9:

Rewrite the Equation The first step is to solve for **n** by rewriting the logarithmic equation in its exponential form.

- The equation is $\log_3(\log_2 n) = N$.
- Rewriting the outer log (base 3) gives: 3^N = $\log_2 n$
- Rewriting the inner log (base 2) gives: **n = 2^(3^N)**

Find Possible Values We are given that **n** and **N** are integers and **1 < n < 1000**. We can now test integer values for **N** to find the corresponding values of **n** that fit within this range.

- If **N = 0**: n = 2^(3^0) = 2^1 = 2. This is a valid solution (1 < 2 < 1000).
- If **N = 1**: n = 2^(3^1) = 2^3 = 8. This is a valid solution (1 < 8 < 1000).
- If **N = 2**: n = 2^(3^2) = 2^9 = 512. This is a valid solution (1 < 512 < 1000).

- If **N = 3**: n = 2^(3^3) = 2^27. This number is much larger than 1000, so it's not a valid solution. *(For any negative integer N, 3^N would be a fraction, and n would not be an integer.)*

Calculate the Sum The only possible integer values for **n** are 2, 8, and 512. The final step is to find their sum.

- Sum = 2 + 8 + 512 = 522 🧠

Problem 10:

The product of all four hidden integers is **840**.

Set Up the Equations Let's call the four hidden positive integers $\mathbf{h_1}$**,** $\mathbf{h_2}$**,** $\mathbf{h_3}$**,** and $\mathbf{h_4}$. According to the rule, the number on the front of each card is the product of the *other three* hidden numbers. This gives us four equations:

- $h_2 \times h_3 \times h_4 = 280$
- $h_1 \times h_3 \times h_4 = 168$
- $h_1 \times h_2 \times h_4 = 105$
- $h_1 \times h_2 \times h_3 = 120$

Multiply the Equations Together The quickest way to solve this is to multiply all four equations together.

- On the left side, each hidden number (h_1, h_2, h_3, h_4) appears three times. So, the product is $(h_1 \times h_2 \times h_3 \times h_4)^3$.
- On the right side, we have the product of the known numbers: $280 \times 168 \times 105 \times 120$.
- This gives us: $(h_1 \times h_2 \times h_3 \times h_4)^3 = 280 \times 168 \times 105 \times 120$

Use Prime Factorization To avoid dealing with huge numbers, let's break down the right side into its prime factors:

- $280 = 2^3 \times 5 \times 7$
- $168 = 2^3 \times 3 \times 7$
- $105 = 3 \times 5 \times 7$
- $120 = 2^3 \times 3 \times 5$ Now, let's combine all these prime factors:
- 2s: $2^3 \times 2^3 \times 2^3 = 2^9$
- 3s: $3 \times 3 \times 3 = 3^3$
- 5s: $5 \times 5 \times 5 = 5^3$

- 7s: $7 \times 7 \times 7 = 7^3$ So, the equation is: $(h_1 \times h_2 \times h_3 \times h_4)^3 = 2^9 \times 3^3 \times 5^3 \times 7^3$

Find the Final Product To find the product of the hidden numbers, we just need to take the cube root of the right side.

- $h_1 \times h_2 \times h_3 \times h_4 = (2^9 \times 3^3 \times 5^3 \times 7^3)^{1/3}$
- $= 2^3 \times 3^1 \times 5^1 \times 7^1$
- $= 8 \times 3 \times 5 \times 7 = 840$ 🧠

Problem 11:

To solve this, we'll find all the perfect squares, cubes, and higher powers within the range of 100 to 400, and then count the unique numbers.

Perfect Squares (x^2) We need to find the squares between 100 and 400.

- $10^2 = 100$ (the first one)
- $20^2 = 400$ (the last one) The squares in this range are from 10^2 to 20^2, which gives us 20 - 10 + 1 = 11 numbers.
- **List:** {100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400}

Perfect Cubes (x^3) Next, we find the cubes between 100 and 400.

- $4^3 = 64$ (too small)
- $5^3 = 125$
- $6^3 = 216$
- $7^3 = 343$
- $8^3 = 512$ (too large)
- **List:** {125, 216, 343}

Higher Powers We also need to check for perfect fourth powers, fifth powers, etc.

- **4th Powers:** $4^4 = 256$
- **5th Powers:** $3^5 = 243$
- **6th Powers:** $3^6 = 729$ (too large)
- **7th Powers:** $2^7 = 128$
- **8th Powers:** $2^8 = 256$
- **List:** {128, 243, 256}

Combine and Count the Unique Numbers Now, let's combine all the numbers we found and remove any duplicates.

- **From Squares:** 100, 121, 144, 169, 196, 225, **256**, 289, 324, 361, 400
- **From Cubes:** 125, 216, 343
- **From Higher Powers:** 128, 243, (**256** is a duplicate)

The final unique list of perfect powers is: {100, 121, 125, 128, 144, 169, 196, 216, 225, 243, 256, 289, 324, 343, 361, 400}

Counting these numbers, we find there are a total of **16**. 🔢

Problem 12:

Analyze the Geometry Let the center of the circle be point **O**. The problem states that the circle is tangent to the line AB at point B and to the line AC at point C.

By the properties of tangents, the radius to the point of tangency is perpendicular to the tangent line. This means OB ⊥ AB and OC ⊥ AC.

This creates two right-angled triangles: ΔABO and ΔACO.

- We know the radius is 1, so **OB = OC = 1**.

Use Triangle Properties

- In the quadrilateral ABOC, we know three of the angles:
  - ∠BAC = 60° (since ABC is an equilateral triangle).
  - ∠ABO = 90° (from tangency).
  - ∠ACO = 90° (from tangency).
- The line segment **AO** connects the vertex A to the center of the circle O, so it must bisect the angle ∠BAC. This means ∠OAB = 60° / 2 = 30°.

Calculate the Side Length Now consider the right-angled triangle **ΔABO**.

We know ∠OAB = 30° and the side opposite this angle, **OB**, has a length of 1.

The side **AB** is the side length of the equilateral triangle, which we need to find.

Using basic trigonometry (tan = opposite / adjacent): tan(30°) = OB / AB

1 / √3 = 1 / AB

Solving for AB gives us **AB = √3**.

Since ABC is an equilateral triangle, all its sides have the same length. 📐

Problem 13:

Set Up the Equations Let the length of the whole wire be 2 meters. We'll call the length of the longer part **L** and the shorter part **S**. Based on the problem, we can create two equations:

- The two parts make up the whole wire: $L + S = 2$
- The ratio relationship: $L / S = 2 / L$

Combine the Equations From the first equation, we can express the shorter part in terms of the longer part: $S = 2 - L$. Now, we can substitute this into the second equation:

- $L / (2 - L) = 2 / L$

Solve for L Cross-multiply to get rid of the fractions:

- $L \times L = 2 \times (2 - L)$
- $L^2 = 4 - 2L$ Rearrange this into a standard quadratic equation ($ax^2 + bx + c = 0$):
- $L^2 + 2L - 4 = 0$

Since this can't be factored easily, we use the quadratic formula to solve for L:

- $L = [-b \pm \sqrt{(b^2 - 4ac)}] / 2a$
- $L = [-2 \pm \sqrt{(2^2 - 4(1)(-4))}] / 2$
- $L = [-2 \pm \sqrt{(4 + 16)}] / 2$
- $L = [-2 \pm \sqrt{20}] / 2$
- $L = [-2 \pm 2\sqrt{5}] / 2 = -1 \pm \sqrt{5}$

Choose the Correct Solution We get two possible answers: $-1 + \sqrt{5}$ and $-1 - \sqrt{5}$. Since length must be a positive value, we choose the positive root.

- $L = \sqrt{5} - 1$ *(This is a classic problem describing the* ***Golden Ratio****.)*

Problem 14:

Convert to Base 10 First, we translate the number N from its representations in base b and base b-2 into our familiar base 10.

- $N = (111)_b = 1\times b^2 + 1\times b^1 + 1\times b^0 = b^2 + b + 1$
- $N = (160)_{b-2} = 1\times(b-2)^2 + 6\times(b-2)^1 + 0\times(b-2)^0 = (b^2 - 4b + 4) + (6b - 12) = b^2 + 2b - 8$

Solve for b Since both expressions are equal to N, we can set them equal to each other to solve for b.

- $b^2 + b + 1 = b^2 + 2b - 8$
- $b + 1 = 2b - 8$
- $9 = b$ The base b is **9**.

Find the Value of N in Base 10 Now we can use the value of b to find the value of N in base 10.

- $N = b^2 + b + 1 = 9^2 + 9 + 1 = 81 + 9 + 1 = 91$ So, **N = 91** in base 10.

Convert N to the New Base The question asks for N in base b-4.

- The new base is 9 - 4 = 5. We need to convert the number 91 from base 10 to base 5. We can do this using division with remainders.
- 91 ÷ 5 = 18 with a remainder of **1**
- 18 ÷ 5 = 3 with a remainder of **3**
- 3 ÷ 5 = 0 with a remainder of **3** Reading the remainders from the bottom up, we get **331**. So, N in base 5 is $\mathbf{(331)_5}$. 🔢

Problem 15:

Find the New Total The class starts with 20 students. After one new boy joins, the total number of students becomes:

- 20 + 1 = 21

Determine the Number of Boys and Girls Let **B** be the new number of boys and **G** be the number of girls. We have two pieces of information about the new class composition:

- The total number of students is 21: B + G = 21
- There are twice as many boys as girls: B = 2G

Now, we can substitute the second equation into the first:

- (2G) + G = 21
- 3G = 21
- G = 7 There are **7 girls**. The number of boys is B = 2 × 7 = 14.

Calculate the Product The question asks for the product of the number of boys and girls.

- 14 (boys) × 7 (girls) = 98 👨‍👩‍👦‍👦