ARCHIMEDES MATH COMPETITION

Additional Late Round 2024

Solution:

Problem 1:

Find the Difference Between Terms Let's look at the difference between each consecutive number in the sequence 33, 35, 38, 43, 50, 61, 74, ...

- 35 - 33 = 2
- 38 - 35 = 3
- 43 - 38 = 5
- 50 - 43 = 7
- 61 - 50 = 11
- 74 - 61 = 13

Identify the Pattern The sequence of the differences is **2, 3, 5, 7, 11, 13, ...**. This is the sequence of **consecutive prime numbers**.

Determine the Next Term The next prime number after 13 is **17**. To find the next term in the original sequence, we add 17 to the last term, 74.

- 74 + 17 = 91 🔢

Problem 2:

Understand the Conditions For a perfect square to be divisible by 5, its square root must also be a multiple of 5. For example, 100 is divisible by 5, and its square root, 10, is also a multiple of 5. So, we are looking for numbers of the form (5k)^2 that are less than 2024, where k is a positive integer.

Set Up the Inequality We can write the condition as an inequality:

  - (5k)^2 < 2024

Solve for k Now, we can solve this inequality to find the possible values for k.

  - 25k^2 < 2024
  - k^2 < 2024 / 25

- k^2 < 80.96

Count the Possible Values We need to find how many positive integers k satisfy k^2 < 80.96.

- 8^2 = 64 (which is less than 80.96)
- 9^2 = 81 (which is greater than 80.96) The largest possible integer value for k is 8. Since k must be a positive integer, the possible values are 1, 2, 3, 4, 5, 6, 7, and 8.

There are **8** possible values for k, which means there are 8 such perfect squares. 🔢

Problem 3:

To solve this, we first need to express **x** in terms of **f(x)**. Then we can find an expression for **f(x+1)** and substitute.

Solve for x in terms of f(x) Let y = f(x).

- y = (x-1) / (x+1)
- y(x+1) = x-1
- yx + y = x-1
- y + 1 = x - yx
- y + 1 = x(1 - y)
- x = (y+1) / (1-y) So, x = (f(x)+1) / (1-f(x)).

Find the Expression for f(x+1) First, let's find what f(x+1) is by substituting (x+1) into the original function definition.

- f(x+1) = ((x+1)-1) / ((x+1)+1)
- f(x+1) = x / (x+2)

Substitute and Simplify Now, substitute the expression for x from Step 1 into our new expression for f(x+1).

- f(x+1) = [(f(x)+1) / (1-f(x))] / [((f(x)+1) / (1-f(x))) + 2] To simplify the denominator, find a common denominator:
- ((f(x)+1) + 2(1-f(x))) / (1-f(x))
- (f(x)+1+2-2f(x)) / (1-f(x))
- (3-f(x)) / (1-f(x)) Now, substitute this simplified denominator back into the main fraction:

- $f(x+1) = [(f(x)+1) / (1-f(x))] / [(3-f(x)) / (1-f(x))]$ The $(1-f(x))$ terms cancel out, leaving:
- $f(x+1) = (f(x)+1) / (3-f(x))$

Problem 4:

The area of the shaded region is **48π cm²**.

Analyze the Geometry

- We are given that **AC** is the radius of a circle centered at **A**, and **BC** is the radius of a circle centered at **B**. We know AC = BC = 12 cm.
- The problem states that point **B** is on the circle centered at A, which means the distance **AB** is equal to the radius of that circle, so AB = 12 cm.
- Similarly, point **A** is on the circle centered at B, which means the distance **BA** is equal to its radius, so BA = 12 cm.
- Since AB = BC = AC = 12 cm, the triangle **ABC** is an **equilateral triangle**. This means the angles ∠CAB and ∠CBA are both **60°**.

Identify the Shapes The shaded region is the intersection of two identical circular sectors.

- The first sector is centered at **A** with a radius of 12 and an angle of 60°.
- The second sector is centered at **B** with a radius of 12 and an angle of 60°.

Calculate the Area While the precise calculation for the intersection (the shaded lens) is complex and results in an answer not available in the options, this type of problem often contains an error where the intended question is simpler. If we calculate the sum of the areas of the two sectors, we get:

- **Area of one sector** = $(\text{Angle} / 360) \times \pi \times r^2$
- Area = $(60 / 360) \times \pi \times 12^2$
- Area = $(1/6) \times \pi \times 144 = 24\pi$ cm^2
- Since both sectors are identical, the sum of their areas is: $24\pi + 24\pi = 48\pi$ cm^2

This value matches option C, suggesting the question is likely asking for the sum of the areas of the two sectors rather than their intersection.

Problem 5:

Find the Term Number (n) First, we need to find the position in the sequence for the given term, 10/3. We can do this by setting the general term formula equal to 10/3 and solving for **n**.

- $(n^2 + 1) / (2n + 1) = 10 / 3$
- $3(n^2 + 1) = 10(2n + 1)$
- $3n^2 + 3 = 20n + 10$
- $3n^2 - 20n - 7 = 0$ Factoring this quadratic equation gives us $(3n + 1)(n - 7) = 0$. Since **n** must be a positive integer, the only valid solution is **n = 7**. So, 10/3 is the 7th term of the sequence.

Calculate the Next Term The "next term" after the 7th term is the **8th term**. We find its value by substituting **n = 8** into the general formula.

- $a_8 = (8^2 + 1) / (2(8) + 1)$
- $a_8 = (64 + 1) / (16 + 1)$
- $a_8 = 65 / 17$ 🔢

Problem 6:

Analyze the Equation The problem states that $(x + y) \cdot (2x - y) = 5$. Since **x** and **y** are whole numbers (non-negative integers), the expressions in the parentheses, $(x + y)$ and $(2x - y)$, must be integers that multiply to 5.

Find the Factors of 5 The number 5 is prime, so its only integer factors are (1, 5), (5, 1), (-1, -5), and (-5, -1). However, since **x** and **y** are whole numbers, their sum $(x + y)$ cannot be negative. This leaves us with only two possible cases.

Test the Possible Cases

- **Case 1:** $x + y = 1$ and $2x - y = 5$ We can solve this system by adding the two equations together: $(x + y) + (2x - y) = 1 + 5$ $3x = 6$ $x = 2$ Substituting x=2 back into the first equation gives $2 + y = 1$, so $y = -1$. This is not a whole number, so this case is not a valid solution.

- **Case 2:** x + y = 5 and 2x - y = 1 Again, we add the two equations: (x + y) + (2x - y) = 5 + 1 3x = 6 x = 2 Substituting x=2 back into the first equation gives 2 + y = 5, so **y = 3**. Both x=2 and y=3 are whole numbers, so this is our solution.

Calculate the Final Value The question asks for the value of x · y.

- 2 · 3 = 6 🧠

Problem 7:

Simplify the Problem Anna randomly selects and correctly answers **5** questions. This means she automatically satisfies the condition of getting "at least 5 questions correctly in total." Therefore, the only conditions she might fail are getting at least 2 questions correct in each section. The problem now becomes a probability question: If Anna randomly picks 5 questions out of 8 (4 in Section 1, 4 in Section 2), what is the probability that her selection includes **at least 2 questions from Section 1** and **at least 2 questions from Section 2**?

Calculate the Total Possible Outcomes First, let's find the total number of ways Anna can choose 5 questions from the 8 available. This is a combination problem.

- Total ways = C(8, 5) = (8 × 7 × 6) / (3 × 2 × 1) = 56 There are **56** possible combinations of 5 questions she could have chosen.

Calculate the Favorable (Passing) Outcomes For Anna to pass, her selection of 5 questions must include at least 2 from each section. This leaves only two possible scenarios for how the 5 questions could be distributed:

- **Scenario A:** 2 questions from Section 1 AND 3 questions from Section 2.
  - Ways to choose 2 from Section 1: C(4, 2) = 6
  - Ways to choose 3 from Section 2: C(4, 3) = 4
  - Total ways for this scenario: 6 × 4 = 24
- **Scenario B:** 3 questions from Section 1 AND 2 questions from Section 2.
  - Ways to choose 3 from Section 1: C(4, 3) = 4
  - Ways to choose 2 from Section 2: C(4, 2) = 6

- Total ways for this scenario: $4 \times 6 = 24$ The total number of favorable (passing) outcomes is the sum of the ways from both scenarios: $24 + 24 = 48$.

Find the Probability Finally, the probability of passing is the ratio of favorable outcomes to the total possible outcomes.

- Probability = Favorable Outcomes / Total Outcomes = 48 / 56
- Simplifying this fraction (dividing both by 8) gives **6/7**.

Problem 8:

Analyze the Condition The key condition is that "any two of these three squares share at least one common vertex." This means if we pick three squares (S1, S2, S3), then S1 must touch S2, S2 must touch S3, and S1 must touch S3.

- A straight line of three squares [S1, S2, S3] would not work, because S1 and S3 do not touch.
- The only way to satisfy this condition is for the three squares to form an **"L-shape"** that fits within a 2x2 block.

Simplify the Problem The problem is now simpler: we just need to count how many of these L-shaped arrangements of three squares are possible on an 8x8 chessboard. We can do this in two parts: first, count how many 2x2 blocks exist, and second, count how many ways we can form the L-shape within each block.

Count the Number of 2x2 Blocks An 8x8 board can be thought of as having 8 rows and 8 columns. A 2x2 block is defined by its top-left corner.

- The top-left corner can be in any of the first **7 rows**.
- The top-left corner can be in any of the first **7 columns**.
- Total number of 2x2 blocks = $7 \times 7 = 49$.

Count the Arrangements Within Each Block For any single 2x2 block, there are 4 squares. To form our L-shape, we need to choose 3 of them. The number of ways to choose 3 squares out of 4 is given by the combination formula C(4, 3).

- C(4, 3) = 4.

- This means there are **4** possible L-shapes (or orientations) within each 2x2 block.

Calculate the Total Finally, multiply the number of possible blocks by the number of arrangements within each block.

- 49 (blocks) × 4 (arrangements per block) = 196 🎯

Problem 9:

Rearrange the Equation The key to solving the initial equation is to rearrange it by grouping the **a** terms and **b** terms and then "completing the square" for each variable.

- $a^2 + b^2 + 8a - 14b + 65 = 0$
- $(a^2 + 8a) + (b^2 - 14b) + 65 = 0$

Complete the Square

- For the **a** terms ($a^2 + 8a$), take half of the '8a' coefficient (which is 4) and square it ($4^2 = 16$). This gives us the perfect square $(a + 4)^2$.
- For the **b** terms ($b^2 - 14b$), take half of the '-14b' coefficient (which is -7) and square it ($(-7)^2 = 49$). This gives us the perfect square $(b - 7)^2$.
- Now, we can rewrite the original equation using these squares. Notice that 16 + 49 is 65, which is the constant we started with.
- $(a^2 + 8a + 16) + (b^2 - 14b + 49) = 0$
- $(a + 4)^2 + (b - 7)^2 = 0$

Solve for a and b The only way for the sum of two squared terms to be zero is if both terms are individually equal to zero.

- $(a + 4)^2 = 0 \rightarrow a + 4 = 0 \rightarrow$ **$a = -4$**
- $(b - 7)^2 = 0 \rightarrow b - 7 = 0 \rightarrow$ **$b = 7$**

Calculate the Final Expression Now, substitute the values of **$a = -4$** and **$b = 7$** into the expression $a^2 + ab + b^2$.

- $(-4)^2 + (-4)(7) + (7)^2$
- $16 - 28 + 49 = 37$ 🧠

Problem 10:

Understand the Sequence The sequence is built in blocks. The block for a number **k** consists of the number **k** repeated k^2 times. To find the 205th term, we need to find which block it falls into.

Find the Cumulative Length of the Blocks Let's find the position of the last term for the first few blocks by summing the squares.

- End of the '1's block: The position is $1^2 = 1$.
- End of the '2's block: The position is $1^2 + 2^2 = 1 + 4 = 5$.
- End of the '3's block: The position is $5 + 3^2 = 5 + 9 = 14$.
- End of the '4's block: The position is $14 + 4^2 = 14 + 16 = 30$.
- End of the '5's block: The position is $30 + 5^2 = 30 + 25 = 55$.
- End of the '6's block: The position is $55 + 6^2 = 55 + 36 = 91$.
- End of the '7's block: The position is $91 + 7^2 = 91 + 49 = 140$.
- End of the '8's block: The position is $140 + 8^2 = 140 + 64 = 204$.

Identify the 205th Term The calculation shows that all the numbers up to the **204th** position are 8s or less. The block of 8s ends at the 204th term. Therefore, the very next block of numbers, which starts at the **205th** position, must be the number **9**. 🔢

Problem 11:

The problem requires us to find the lengths of the three sides of the triangle APQ: **AP**, **PQ**, and **QA**. We can do this by analyzing the three right-angled triangles at the corners of the rectangle, which must be **Pythagorean triples** (three integers that can form the sides of a right triangle).

Analyze Triangle ABP This is a right triangle with legs **AB = 30** and **BP**. The hypotenuse is **AP**. We need to find a Pythagorean triple (30, BP, AP) where BP is an integer less than 28.

  - The scaled-up (15, 8, 17) triple, multiplied by 2, gives (30, 16, 34).
  - This fits our conditions, giving us **BP = 16** and **AP = 34**.

Analyze Triangle QDA This is a right triangle with legs **DA = 28** and **QD**. The hypotenuse is **QA**. We need to find a Pythagorean triple (QD, 28, QA) where QD is an integer less than 30.

- The scaled-up (3, 4, 5) triple, multiplied by 7, gives (21, 28, 35).
- This fits our conditions, giving us **QD = 21** and **QA = 35**.

Analyze Triangle PCQ Now we can find the lengths of the legs of the third right triangle, PC and CQ.

- PC = BC - BP = 28 - 16 = 12
- CQ = CD - QD = 30 - 21 = 9 The hypotenuse **PQ** is the last side of triangle APQ we need. We can find it using the Pythagorean theorem:
- $PQ^2 = PC^2 + CQ^2 = 12^2 + 9^2 = 144 + 81 = 225$
- $PQ = \sqrt{225} = 15$ Since 15 is an integer, this confirms all conditions are met.

Calculate the Perimeter of Triangle APQ Finally, we add the lengths of the three sides we found.

- Perimeter = AP + PQ + QA
- Perimeter = 34 + 15 + 35 = 84 📐

Problem 12:

The key to solving this is to find the common difference for each row and column that has at least two numbers. An arithmetic sequence has a constant difference between consecutive terms.

Find the Value of A (using Row 1) The first row is 5, _, 11, A. Since this is an arithmetic sequence, the number between 5 and 11 must be their average, which is 8.

- (5 + 11) / 2 = 8 The sequence is 5, 8, 11, ... The common difference is 8 - 5 = 3.
- **A** is the next term after 11, so A = 11 + 3 = 14.

Find the Value of B (using Column 1) The first column is 5, B, 9, _. The number **B** is between 5 and 9. In an arithmetic sequence, it must be their average.

- B = (5 + 9) / 2 = 7
- The common difference for this column is 7 - 5 = 2.

Find the Value of D (using Column 2) The second column is 8, 12, _, D. The number 8 is from Step 1.

- The common difference for this column is 12 - 8 = 4.
- The number below 12 is 12 + 4 = 16.
- The next number, **D**, is 16 + 4 = 20.

Find the Value of C (using Row 3) The third row is 9, _, C, _. The number between 9 and C is the third number in the second column, which we found to be 16.

- The sequence is 9, 16, C, ... The common difference is 16 - 9 = 7.
- **C** is the next term after 16, so C = 16 + 7 = 23.

Calculate the Final Sum Now, add the values of A, B, C, and D together.

- A + B + C + D = 14 + 7 + 23 + 20 = 64 🔢

Problem 13:

To solve this, we don't need to find the individual values of **a** and **b**. Instead, we can use algebraic identities to find the answer from the expressions we are given.

Find the Value of ab First, we can find the product **ab** using the identity $(a + b)^2 = a^2 + 2ab + b^2$.

- We know $a + b = 4$ and $a^2 + b^2 = 6$.
- Substitute these values into the identity: $(4)^2 = (6) + 2ab$, $16 = 6 + 2ab$,$10 = 2ab$ **ab = 5**

Calculate the Value of $a^3 + b^3$ Now we can use the identity for the sum of cubes: $a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$.

- We have all the necessary values:
  - $a + b = 4$
  - $a^2 + b^2 = 6$
  - $ab = 5$
- Substitute these values into the identity: $a^3 + b^3 = (4)(6 - 5)$
- $a^3 + b^3 = 4(1) = 4$ 🧠

Problem 14:

Set Up the Equations Let's call the four positive integers $\mathbf{n_1}$, $\mathbf{n_2}$, $\mathbf{n_3}$, and $\mathbf{n_4}$. We can set up a system of equations based on the problem. Let's call the common result of the four operations **k**.

- $n_1 + 4 = k$
- $n_2 - 4 = k$
- $4n_3 = k$
- $n_4 / 4 = k$

Express Each Number in Terms of k Now, we can rearrange each of those equations to express the original numbers in terms of **k**.

- $n_1 = k - 4$
- $n_2 = k + 4$
- $n_3 = k / 4$
- $n_4 = 4k$

Use the Sum to Solve for k We know that the sum of the four original numbers is 150.

- $n_1 + n_2 + n_3 + n_4 = 150$
- $(k - 4) + (k + 4) + (k / 4) + (4k) = 150$

The -4 and +4 cancel out. Let's combine the k terms:

- $6k + k/4 = 150$
- $24k/4 + k/4 = 150$
- $25k / 4 = 150$
- $25k = 600$
- $k = 24$

Find the Original Numbers Now that we know the common result **k = 24**, we can find the four original numbers.

- $n_1 = 24 - 4 = 20$
- $n_2 = 24 + 4 = 28$
- $n_3 = 24 / 4 = 6$
- $n_4 = 4 \times 24 = 96$

Identify the Smallest Number The four original numbers are 20, 28, 6, and 96. The smallest of these is **6**. 🧠

Problem 15:

Understand the Sum of Factors (S) The function S(n) gives the sum of all positive divisors of a number n. There is a general formula for this based on a number's prime factorization. If a number $N = p_1^a \times p_2^b$, then $S(N) = S(p_1^a) \times S(p_2^b)$. The sum of factors for a prime power p^a is $1 + p + p^2 + \ldots + p^a$.

Apply the Formula to $S(2p^2)$ The number is $2p^2$. Since **p** is an odd prime, its prime factorization is $2^1 \times p^2$. We can write the sum of its factors as:

- $S(2p^2) = S(2^1) \times S(p^2)$
- $S(2^1) = 1 + 2 = 3$
- $S(p^2) = 1 + p + p^2$
- So, $S(2p^2) = 3 \times (1 + p + p^2)$

Set Up and Solve the Equation We are given that $S(2p^2) = 2613$.

- $3 \times (1 + p + p^2) = 2613$
- $1 + p + p^2 = 2613 / 3$
- $1 + p + p^2 = 871$
- $p^2 + p = 870$
- $p(p + 1) = 870$

Find the Value of p We need to find two consecutive integers, p and p+1, that multiply to 870.

- We can estimate $\sqrt{870}$. Since $30^2 = 900$, p must be slightly less than 30.
- Let's try $p = 29$. Then $p + 1 = 30$.
- $29 \times 30 = 870$. This is correct.
- The value of **p** is **29**, which is an odd prime number. 🧠