Solution:

Problem 1:

The area of the square ABCD is **106**.

Let the radius of the circumcircle be R and its center be at the origin (0,0). The area of the inscribed square is 2R2.

Let the vertices of the square be A(–R,0), C(R,0), B(0,R), and D(0,–R). Let the point P on the circle have coordinates (x,y), so $x^{2+}y^2 = R^2$.

We are given $PA \cdot PC = 56$. We can write the square of this product in terms of the coordinates:

$$(PA \cdot PC)^2 = ((x+R)^2 + y^2)((x-R)^2 + y^2)$$

Using $x^2 + y^2 = R^2$, this simplifies:

$$(PA \cdot PC)^2 = (2R^2 + 2Rx)(2R^2 - 2Rx) = 4R^4 - 4R^2x^2$$

So, $4R^4 - 4R^2x^2 = 56^2 = 3136$. Dividing by 4 gives:

$$R^4 - R^2x^2 = 784$$

Similarly, for $PB \cdot PD = 90$:

$$(PB \cdot PD)^2 = (x^2 + (y-R)^2)(x^2 + (y+R)^2)$$

$$(PB \cdot PD)^2 = (2R^2 - 2Ry)(2R^2 + 2Ry) = 4R^4 - 4R^2y^2$$

So,
$4R^4 - 4R^2x^2 = 56^2 = 3136.$ Dividing by 4 gives:

$$R^4 - R^2y^2 = 2025$$

Now, add the two equations:

$$(R^4 - R^2x^2) + (R^4 - R^2y^2) = 784 + 2025$$

$$2R^4 - R^2(x^2 + y^2) = 2809$$

Since, $x^2 + y^2 = R^2$, the equation becomes:

$$2R^4 - R^2(R)^2 = 2809$$

$$R^4 = 2809$$

Solving for R^2: $R^2 = 53$

Finally, the area of the square is:

Area=$2R^2 = 2 \times 53 = 106$

Problem 2:

Hoppy the Frog can reach the ground floor in **21** different ways.

This classic problem can be solved by noticing a pattern similar to the Let's find the number of ways, let's call it W(n), for Hoppy to climb n stairs.To reach any given stair, Hoppy must have landed on it from either the stair below it (a 1-stair hop) or from two stairs below (a 2-stair hop). This means the total number of ways to reach a stair is the sum of the ways to reach the two preceding stairs.

The relationship is: W(n)=W(n−1)+W(n−2)

Let's calculate the number of ways step-by-step:

1 stair: 1 way (a single 1-stair hop)

2 stairs: 2 ways (1-1 or 2)

3 stairs: W(2)+W(1)=2+1=3 ways (1-1-1, 1-2, or 2-1)

4 stairs: W(3)+W(2)=3+2=5 ways

5 stairs: W(4)+W(3)=5+3=8 ways

6 stairs: W(5)+W(4)=8+5=13 ways

7 stairs: W(6)+W(5)=13+8=21 ways

So, Hoppy has 21 unique paths to get from the basement to the ground floor.

Problem 3:

The sum of all real numbers m for which the expressions are equal is **7**.

To find the values of m, we first set the two expressions equal to each other:

$$2^m \cdot \sqrt{\frac{1}{4096}} = 2 \cdot \sqrt[m]{\frac{1}{4096}}$$

First, let's simplify the number 4096. It is a power of 2:

$$4096 = 2^{12}$$

Therefore, $\frac{1}{4096} = 2^{-12}$.

Now, substitute this into the equation:

$$2^m \cdot \sqrt{2^{-12}} = 2 \cdot \sqrt[m]{2^{-12}}$$

Next, we simplify the roots using the exponent rule:

- $\sqrt{2^{-12}} = (2^{-12})^{\frac{1}{2}} = 2^{-6}$
- $\sqrt[m]{2^{-12}} = (2^{-12})^{\frac{1}{m}} = 2^{-\frac{12}{m}}$

$$2^{m-6} = 2^{1-\frac{12}{m}}$$

Since the bases are equal, we can set the exponents equal to each other:

$$m - 6 = 1 - \frac{12}{m}$$

Now, we solve this equation for m.

$$m^2 - 7m + 12 = 0$$

We can solve this quadratic equation by factoring. We look for two numbers that multiply to 12 and add up to -7.

These numbers are -3 and -4. (m−3)(m−4)=0 The solutions for m are m=3 and m=4.

The problem asks for the sum of all such real numbers m. Sum = 3+4=7.

Problem 4:

The value of a+b is **2023**.

Solution

This problem involves a sum that can be simplified by recognizing it as a telescoping series. Let's look at the general term of the sum.

The general term is $\frac{k}{(k+1)!}$. The key insight is to rewrite the numerator, k, as (k+1)-1.

This allows us to split the term into two parts:

$\frac{k}{(k+1)!} = \frac{(k+1)-1}{(k+1)!} = \frac{k+1}{(k+1)!} - \frac{1}{(k+1)!}$.

Since $(k+1)! = (k+1) \cdot k!$

So, the general term becomes the difference of two consecutive terms:

$$\frac{k}{(k+1)!} = \frac{1}{k!} - \frac{1}{(k+1)!}$$

Now we can write out the entire sum:

$$S = \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \cdots + \left(\frac{1}{2021!} - \frac{1}{2022!}\right)$$

As you can see, the second part of each term cancels out the first part of the next term. This leaves only the very first and very last terms.

$$S = \frac{1}{1!} - \frac{1}{2022!}$$

Since $1! = 1$, the sum is:

$$S = 1 - \frac{1}{2022!}$$

We are given that the sum can be expressed as $a - \frac{1}{b!}$. By comparing our result with this form, we can see that:

- $a = 1$
- $b = 2022$

The problem asks for the value of $a + b$.

$$a + b = 1 + 2022 = 2023$$

Problem 5:

The perimeter of the rhombus is **3125/11**.

Find the rhombus's height. The distance between the parallel sides (DA and BC) is the diameter of the incircle. You find it by simply adding the two given distances: 9 + 16 = 25. This gives you the radius, r = 12.5.

Find the rhombus's angle. The other two distances (5 and 9) tell you exactly where the point P is relative to the corner A. This relationship lets you calculate the cosine of the rhombus's acute angle (∠BAD), which turns out to be 117/125.

Calculate the perimeter. With the height (2r) and the angle, you can find the length of one side. Once you have the side length, just multiply it by four to get the perimeter.

Find the Inradius (r): The diameter of the incircle (2r) is the distance between the parallel sides DA and BC. $2r=9+16=25 \Rightarrow r=12.5$

Find the Rhombus Angle (2α): The distances from P to the adjacent sides AB (5) and DA (9) allow us to find cos(2α). This is done by relating the distances to the angles formed by the point P and the normals to the sides.

$$\cos(2a) = \frac{117}{125}$$

From this, we find the sine of the angle:

$$\sin(2a) = \sqrt{1 - (\frac{117}{125})^2} = \frac{44}{125}$$

Find the Perimeter: The side length (s) is found using the formula

s = 2r / sin(2α).

$$s = \frac{25}{44/125} = \frac{3125}{44}$$

The perimeter is **4s**: Perimeter=$4 \times \frac{3125}{44} = \frac{3125}{11}$

Problem 6:

There are **315** such integers.

The core idea is to use the **Principle of Inclusion-Exclusion**. First, count all the numbers divisible by 5. Then, from that group, subtract the numbers that are also divisible by 7 or 11. Since this process subtracts numbers divisible by 5, 7, *and* 11 twice, you have to add that small group back in once to get the correct total

The number of integers can be found with one calculation:

(Divisible by 5) - (Divisible by 5 & 7) - (Divisible by 5 & 11) + (Divisible by 5 & 7 & 11)

$$\left\lfloor\frac{2017}{5}\right\rfloor - \left\lfloor\frac{2017}{35}\right\rfloor - \left\lfloor\frac{2017}{55}\right\rfloor + \left\lfloor\frac{2017}{385}\right\rfloor$$

$$403 - 57 - 36 + 5 = \mathbf{315}$$

Problem 7:

The difference between the largest and smallest of the three integers is **236**.

This problem can be solved efficiently with a simple system of equations.

Let the three positive integers be x, y, and z. To make things easier, let's order them from smallest to largest: x ≤ y ≤ z.

When we add these integers in pairs, the sums will correspond to their sizes:

The sum of the two smallest integers will be the smallest sum:

$\boldsymbol{x + y = 998}$

The sum of the two largest integers will be the largest sum:

$\boldsymbol{y + z = 1234}$

The middle sum will be the sum of the smallest and largest:

$x + z = 1050$

The question asks for the difference between the largest and smallest integers, which is $z - x$. We can find this directly by subtracting the equation for the smallest sum from the equation for the largest sum.

$$(y + z) - (x + y) = 1234 - 998$$

The y terms cancel out, leaving us with the desired difference:

$$z - x = 236$$

Problem 8:

The value of A+B+C is **24**.

The core idea is to use the divisibility rules to narrow down the possibilities.

Since the number is a **perfect square** and is divisible by **9 and 4**, its square root must be divisible by **6**.

The number is between 12000 and 12999, which means its square root is between 110 and 114.

You just need to find the only number in that small range that is divisible by 6.

$$12000 \leq k^2 \leq 12999 \Longrightarrow 110 \leq k \leq 114$$

The integer k must be a multiple of 6. The only multiple of 6 in the range {110, 111, 112, 113, 114} is **114**.

The number is 114^2=12996. So, A=9, B=9, and C=6. The sum is A+B+C=9+9+6=24.