ARCHIMEDES MATH COMPETITION

Second Round 2022-2023

Solution:

Problem 1:

The value of j + k is **881**.

Here is the brief solution:

The first equation, $\sqrt{\log_b n} = \log_b \sqrt{n}$, simplifies to $\log_b n = 4$. This means $\mathbf{n = b^4}$.

The second equation, $b \log_b n = \log_b bn$, simplifies to $b(\log_b n) = 1 + \log_b n$.

Substituting the result from the first equation ($\log_b n = 4$) into the second gives $4b = 1 + 4$, which means $\mathbf{b = 5/4}$.

Now we find n: $n = b^4 = (5/4)^4 = 625 / 256$.

Since 625 and 256 are relatively prime, $j = 625$ and $k = 256$. $j + k = 625 + 256 = 881$.

The correct answer: 881

Problem 2:

There are **16** extra-distinct positive integers less than 1000.

For a number n to have five distinct remainders when divided by 2, 3, 4, 5, and 6, the set of remainders must be **{0, 1, 2, 3, 4}**, since the remainder when dividing by 5 cannot be greater than 4.

These remainders are linked. For example, n mod 2 is determined by n mod 6. Analyzing these dependencies reveals that there is only one possible set of congruences:

$n \equiv 0 \pmod{2}$

$n \equiv 1 \pmod{3}$

$n \equiv 2 \pmod{4}$

$n \equiv 3 \pmod{5}$

$n \equiv 4 \pmod{6}$

The smallest positive integer that satisfies all these conditions is **58**. The solutions repeat every LCM(2,3,4,5,6) = 60. So, all extra-distinct numbers are of the form 60k + 58.

To find how many of these are less than 1000, we solve: 60k + 58 < 1000 → 60k < 942 → k < 15.7. The possible values for k are 0, 1, 2, ..., 15, which is a total of **16** numbers.

The correct answer: 16

Problem 3:

There are 14 people (5 men, 9 women) arranged in a circle, forming 7 airs of opposite positions:

The total number of ways to choose 5 positions for the men out of 14 is:

$$N_{total} = \binom{14}{5} = \frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 2002$$

For every man to face a woman, no two men can be opposite each other. This means each man must be in a different pair of opposite spots.

- First, choose 5 of the 7 opposite pairs: $\binom{7}{5} = 21\ ways$
- In each of the 5 chosen pairs, pick one of the two spots for a man: $2^5 = 32$ ways. The number of favorable arrangements is:

$$N_{favorable} = \binom{7}{5} \cdot 2^5 = 21 \cdot 32 = 672$$

The probability is the ratio of favorable to total arrangements:

P=672/2002=48/143

So, m=48 and n=143. They are relatively prime.

m+n=48+143=191

The correct answer: 191

Problem 4:

Assuming the polynomial is $p(x) = x^3 + ax^2 + bx + c$ to correct a typo in the prompt, the problem reduces to finding the number of valid coefficient sets.

The condition that p(m)=p(2) for a unique integer $m \neq 2$ leads to a quadratic equation in m. This quadratic must have exactly one integer solution other than 2. This happens for **18** specific pairs of coefficients (a,b):

- **8** pairs result in a single, repeated integer root.
- **10** pairs result in two distinct integer roots, one of which is 2.

Since the coefficient c can be any of the **41** integers from -20 to 20, the total number of polynomials is:

18 (pairs)×41 (choices for c)=738

The number of such cubic polynomials is **738**.

Problem 5:

The equally spaced integers form an arithmetic progression with a common difference d. Using the given ranges for a1 ([1,10]) and a15 ([241,250]), we can determine that the only possible integer value for the common difference is **d=17**.

Plugging d=17 back into the conditions for a2 and a15, we find the first term must be **a1=3**.

Finally, we calculate a14:

a14=a1+13d=3+13(17)=3+221=224

The sum of the digits of 224 is 2+2+4=8.

Problem 6:

Let the number of apples on the 6 trees form an arithmetic sequence.

Let the **least** number of apples be aaa,

and the **common difference** be ddd.

Then the apples on the trees are:

a, a+d, a+2d, a+3d, a+4d, a+5da,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ a + 5da, a+d, a+2d, a+3d, a+4d, a+5d

The **greatest** number is a+5da + 5da+5d.

Now find the total number of apples:

Total=a+(a+d)+(a+2d)+(a+3d)+(a+4d)+(a+5d)=6a+15d

We're told the total is 990:

9a=990⇒a=110

Now find the greatest number:

Greatest=a+5d=110+5·1105=110+110=220

The correct answer is 220.

Problem 7:

Current exponents:

- 2^5 ⟶needs 1 more to become 2^6
- 3^4 ⟶needs 2 more to become 3^6
- 5^2 ⟶needs 1 more to become 5^3

- 7^2 ⟶needs 1 more to become 7^3

So we need to multiply by:

$$2^1 \times 3^2 \times 5^1 \times 7^1$$

Now compute: $k = 2 \times 9 \times 5 \times 7 = 630$. The correct answer: 630

Problem 8:

The positive odd numbers are:

1, 3, 5, 7, 9, …, k1,\ 3,\ 5,\ 7,\ 9,\ \ldots,\ k1, 3, 5, 7, 9, …, k

These form an **arithmetic sequence** with:

- First term = 1
- Common difference = 2

We are told the sum of these numbers is 15376.

The sum of the first nnn odd numbers is:

$$Sum = n^2$$
$$n = \sqrt{15376} = 124$$

The nth odd number is:

K=2n-1=2(124)-1=248-1=247

The correct answer: 247