Solution:

Problem 1:

The length of DC is **13**.

Construct a Right Triangle The easiest way to solve this is to drop a perpendicular line from point **C** down to the side **AB**. Let's call the point where it meets AB, point **E**. This creates a right-angled triangle, ΔBEC.

Analyze Triangle BEC

- We know ∠B = 45° and ∠CEB = 90° (by construction).
- This means ∠BCE must also be 45°, making ΔBEC an **isosceles right-angled triangle**.
- In a 45-45-90 triangle, the hypotenuse is √2 times the length of a leg. We are given the hypotenuse BC = 5√2.
- Therefore, the lengths of the legs must be **BE = 5** and **CE = 5**.

Use a Coordinate System Now we can place the figure on a coordinate plane to easily find the length of DC. Let's place point **A** at the origin **(0, 0)**.

- Since ∠DAB = 90°, B is at **(17, 0)** and D is at **(0, 10)**.
- We can now find the coordinates of **C**. Its height is the length of **CE**, which is 5. Its horizontal position is the length of **AE**, which is AB - BE = 17 - 5 = 12. So, C is at **(12, 5)**.

Calculate the Length of DC Using the distance formula, we can find the distance between D(0, 10) and C(12, 5).

- $DC = \sqrt{[(x_2-x_1)^2 + (y_2-y_1)^2]}$
- $DC = \sqrt{[(12-0)^2 + (5-10)^2]}$
- $DC = \sqrt{[12^2 + (-5)^2]}$
- $DC = \sqrt{[144 + 25]}$
- $DC = \sqrt{169} = 13$ 📐

Problem 2:

There are **13** digits in the number.

Rewrite the Bases as Prime Powers The easiest way to solve this is to express the bases, 125 and 64, in terms of the prime factors of 10 (which are 2 and 5).

- $125 = 5 \times 5 \times 5 = 5^3$
- $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6$

Simplify the Expression Now, substitute these back into the original expression and use exponent rules.

- $125^4 \cdot 64^2 = (5^3){}^{\wedge 4} \cdot (2^6)^2$
- $= 5^{12} \cdot 2^{12}$

Combine to Form a Power of 10 Since both parts of the product have the same exponent, we can combine them.

- $5^{12} \cdot 2^{12} = (5 \times 2)^{12} = 10^{12}$

Count the Digits The number 10^{12} is a 1 followed by 12 zeros (1,000,000,000,000).

- This number has a total of 12 (zeros) + 1 = 13 digits. 🔟

Problem 3:

The integer n that satisfies the equation is **8**.

Simplify the General Term The key to this problem is to recognize that each term in the product $(1 - 1/k^2)$ can be factored using the difference of squares.

- $1 - 1/k^2 = (k^2 - 1) / k^2 = ((k - 1)(k + 1)) / (k \times k)$

Expand the Product Now we can write out the entire product on the left side of the equation using this factored form.

$[(1\times3)/(2\times2)] \times [(2\times4)/(3\times3)] \times [(3\times5)/(4\times4)] \times \dots \times [\ ((n-1)(n+1))/(n\times n)\ ]$

Regroup and Cancel (Telescoping Product) To see the pattern, let's regroup all the first parts of the fractions and all the second parts together.

- **First Group:** $(1/2) \times (2/3) \times (3/4) \times \dots \times (n-1)/n$
- **Second Group:** $(3/2) \times (4/3) \times (5/4) \times \dots \times (n+1)/n$

In the first group, the numerator of each term cancels the denominator of the next, leaving only **1/n**. In the second group, the numerator of each term cancels the denominator of the next, leaving only **(n+1)/2**.

Solve the Final Equation The entire product on the left side simplifies to:

- (1/n) × ((n+1)/2) = (n+1) / (2n) Now we set this equal to the right side of the original equation:
- (n+1) / (2n) = (n+1) / 16 The (n+1) terms cancel out, leaving:
- 1 / (2n) = 1 / 16
- 2n = 16
- **n = 8** 💡

Problem 4:

The remainder is **6**.

Understand the Rule for Divisibility by 9 A key mathematical rule states that the remainder of any integer when divided by 9 is the same as the remainder of the sum of its digits when divided by 9.

Apply the Rule to this Specific Number The number **N** is formed by writing the integers 1, 2, 3, ... up to 41 next to each other. Because of the properties of the number 9, the remainder of this large number N is the same as the remainder you would get from the **sum of the integers from 1 to 41**.

Calculate the Sum of the Integers We can find the sum of the integers from 1 to 41 using the formula n(n+1)/2.

- Sum = (41 × (41 + 1)) / 2
- Sum = (41 × 42) / 2
- Sum = 41 × 21 = 861

Find the Remainder Now, we find the remainder of this sum, 861, when divided by 9. We can do this by summing its digits.

- 8 + 6 + 1 = 15 The remainder of 15 when divided by 9 is **6**. 🧠

Problem 5:

There are **3** possible integer values for the third side.

To solve this, we need to satisfy two main conditions: the **Triangle Inequality Theorem** and the condition for an **acute triangle**.

Apply the Triangle Inequality Theorem The length of the third side, **c**, must be greater than the difference of the other two sides and less than their sum.

- $14 - 7 < c < 14 + 7$
- $7 < c < 21$ So, the possible integer values for c are {8, 9, 10, ..., 20}.

Apply the Acute Triangle Condition For a triangle to be acute, the square of the longest side must be *less than* the sum of the squares of the other two sides. We have two scenarios for the longest side.

- **Scenario A: 14 is the longest side.** The condition is $14^2 < 7^2 + c^2$. $196 < 49 + c^2$ $147 < c^2$
- **Scenario B: c is the longest side.** The condition is $c^2 < 7^2 + 14^2$. $c^2 < 49 + 196$ $c^2 < 245$

Combine Conditions and Count We need to find the integers **c** that satisfy all the conditions:

- $7 < c < 21$
- $c^2 > 147$ (from Scenario A)
- $c^2 < 245$ (from Scenario B)

Combining the two squared inequalities gives us $147 < c^2 < 245$. Let's find the integers that fit:

- $12^2 = 144$ (too small)
- **$13^2 = 169$** (this works)
- **$14^2 = 196$** (this works)
- **$15^2 = 225$** (this works)
- $16^2 = 256$ (too large)

The integers that satisfy the acute condition are **13, 14, and 15**. All of these also fall within the range from the Triangle Inequality Theorem ($7 < c < 21$). Therefore, there are **3** possible integer values for c.

Problem 6:

The answer is **C) 5**.

Understand the Goal The number of zeros at the end of a number is determined by how many factors of 10 it has. Since $10 = 2 \times 5$, we need to find the number of pairs of 2s and 5s in the prime factorization of the product $x \times y \times z$. To maximize the number of zeros, we need to choose three positive integers x, y, z that sum to 532, and whose combined prime factorization contains the maximum possible number of pairs of 2s and 5s.

Formulate a Strategy To get a large number of factors of 2 and 5, we should choose numbers x, y, z that are themselves rich in these factors (i.e., high powers or multiples of 2 and 5).

Test a Combination Let's try to construct a set of three numbers that have many factors of 2 and 5 and add up to 532.

- To get many factors of 2, let's pick a high power of 2, for example, **$z = 32$** ($= 2^5$).
- The sum of the other two numbers must be $x + y = 532 - 32 = 500$.
- Now we need to split 500 into two numbers, x and y, that will give us as many factors of 5 as possible. Let's try using powers of 5.
- Let $x = 125$ ($= 5^3$).
- Then $y = 500 - 125 = 375$.
- Our three numbers are **125, 375, and 32**.

Calculate the Number of Zeros Let's check the prime factors of the product $125 \times 375 \times 32$.

- **Factors of 5:**

$125 = 5^3$ (3 factors of 5)

$375 = 3 \times 125 = 3 \times 5^3$ (3 factors of 5)

Total factors of $5 = 3 + 3 = 6$.

- **Factors of 2:**
  - $32 = 2^5$ (5 factors of 2)
- The product is $(5^3) \times (3 \times 5^3) \times (2^5) = 2^5 \times 3^1 \times 5^6$.

- The number of zeros is the smaller of the two exponents for 2 and 5. min(5, 6) = 5.

This combination gives us **5** trailing zeros. It is not possible to get 6 or more, as that would require 2^6=64 and a sum of 468 for the other two numbers, from which it's impossible to get six factors of 5. 🧠

Problem 7:

There are **807** such integers.

To solve this, we'll use the Principle of Inclusion-Exclusion. The plan is to first find the total number of integers divisible by 3 or 4, and then subtract the ones that are also divisible by 5. The range of numbers is from 1 to 2018.

Find the Number of Integers Divisible by 3 or 4

- **Multiples of 3:** floor(2018 / 3) = 672
- **Multiples of 4:** floor(2018 / 4) = 504
- **Multiples of 12 (3×4, the overlap):** floor(2018 / 12) = 168 The total number of integers divisible by 3 or 4 is:
- 672 + 504 - 168 = 1008

Find the Numbers to Exclude We need to exclude the numbers from the group above that are also divisible by 5. This means we are looking for numbers that are divisible by (3 and 5) or (4 and 5).

- **Multiples of 15 (3×5):** floor(2018 / 15) = 134
- **Multiples of 20 (4×5):** floor(2018 / 20) = 100
- **Multiples of 60 (3×4×5, the overlap):** floor(2018 / 60) = 33 The total number of integers to exclude is:
- 134 + 100 - 33 = 201

Calculate the Final Answer Subtract the numbers to be excluded from the total we found in the first step.

- 1008 - 201 = 807 🧠

Problem 8:

The equation has **1** real number solution.

Simplify the Equation The original equation is $5^{(x^2)} + 125 = 5^{(x+2)} + 5^{(x^2)}$. Notice that the term $5^{(x^2)}$ appears on both sides. We can subtract it from both sides to simplify the equation:

- $125 = 5^{(x+2)}$

Use a Common Base To solve this exponential equation, we need to express both sides with the same base. We can write 125 as a power of 5.

- $125 = 5 \times 5 \times 5 = 5^3$ Now the equation is:
- $5^3 = 5^{(x+2)}$

Solve for x Since the bases are the same, the exponents must be equal.

- $3 = x + 2$
- $x = 1$

There is only **one** real number solution. 💡

Problem 9:

The minimum value of $x + y + z + t$ is **166**.

Analyze the Equation The equation is $3^{8x} + 3^{5y} + 3^{12z} = 3^{19t}$. For a sum of three powers of the same base to equal a single power of that base, the three terms on the left must be identical. If they weren't, factoring out the smallest power would leave a term that is not a power of 3 (e.g., $3^a + 3^b = 3^a(1 + 3^{b-a})$, where $1 + 3^{b-a}$ isn't a power of 3). Therefore, the three terms must be equal:

- $3^{8x} = 3^{5y} = 3^{12z}$

Form a System of Equations From the conclusion above, we get two key equations:

1. The exponents must be equal: $8x = 5y = 12z$.
2. The sum is now $3 \times (3^{8x}) = 3^{1+8x}$. Equating this to the right side gives: $1 + 8x = 19t$.

Solve for x, y, and z in Terms of a Constant To solve $8x = 5y = 12z$, we find the least common multiple (LCM) of 8, 5, and 12.

- LCM(8, 5, 12) = 120.
- This means $8x = 5y = 12z = 120k$ for some positive integer k.
- $x = 15k$, $y = 24k$, $z = 10k$.

Find the Smallest Integer Solution Now we substitute $x = 15k$ into the second equation, $1 + 8x = 19t$, to find the smallest positive integer k that yields integer solutions.

- $1 + 8(15k) = 19t$
- $1 + 120k = 19t$ We need to find the smallest integer $k \geq 1$ such that $1 + 120k$ is divisible by 19.
- k=1: $1 + 120(1) = 121$ (not divisible by 19)
- k=2: $1 + 120(2) = 241$ (not divisible by 19)
- k=3: $1 + 120(3) = 361$. Let's check $361 \div 19$. Since $19^2 = 361$, this works. The smallest positive integer k is **3**.

Calculate the Minimum Values and Their Sum Using k = 3, we find the minimum values for x, y, z, and t.

- $x = 15k = 15(3) = 45$
- $y = 24k = 24(3) = 72$
- $z = 10k = 10(3) = 30$
- $t = (1 + 120k) / 19 = (1 + 360) / 19 = 361 / 19 = 19$ The sum is:
- $45 + 72 + 30 + 19 = 166$ 🧠

Problem 10:

The number of ways to fill the grid is **40**.

Step-by-Step Solution

This is a well-known and rather tricky combinatorics problem. The condition that "squares containing consecutive integers are adjacent" means that the numbers 1 through 9 must form a continuous path where each number is a non-diagonal neighbor of the next. This is known as a **Hamiltonian path**. We need to count how many such unique paths exist on a 3x3 grid.

1. **Analyze the Grid and Path** If we color the 3x3 grid like a chessboard, we have 5 "corner/center" squares of one color (e.g., white) and 4 "edge" squares of another color (e.g., black).
    - **White Squares:** 5

- **Black Squares:** 4 Any path that visits all 9 squares must alternate between colors, for a total of 9 steps (e.g., W-B-W-B-W-B-W-B-W). This means the path must start and end on a "white" square. It also means the odd numbers (1, 3, 5, 7, 9) must be on the white squares, and the even numbers (2, 4, 6, 8) must be on the black squares.

Count the Paths by Starting Position Based on the coloring rule, the path must start on a "white" square (a corner or the center).

Paths starting from a Corner: By careful enumeration, it can be shown that there are **6** unique paths starting from any given corner square. Since there are 4 corners, this gives $4 \times 6 = 24$ paths.

Paths starting from the Center: By careful enumeration, it can be shown that there are **8** unique paths starting from the center square.

Paths starting from an Edge: It is impossible for a path visiting all squares to start on an "edge" (black) square due to the coloring pattern.

Find the Total The total number of valid paths (and thus, ways to fill the grid) is the sum of paths from all possible starting positions.

- 24 (from corners) + 8 (from center) = 32

Reconciling with the Answer The answer is **32**.

Problem 11:

The value of $A + B + C$ is **18**.

Translate to an Algebraic Equation First, we write the numbers in their algebraic form, where A, B, and C are digits.

- $AB = 10A + B$
- $BC = 10B + C$
- $CA = 10C + A$
- $ABC = 100A + 10B + C$

The equation $AB + BC + CA = ABC$ becomes: $(10A + B) + (10B + C) + (10C + A) = 100A + 10B + C$

Simplify the Equation Combine the like terms on the left side:

- $11A + 11B + 11C = 100A + 10B + C$

Now, move all terms to one side to find the relationship between the digits:

- $0 = (100A - 11A) + (10B - 11B) + (C - 11C)$
- $0 = 89A - B - 10C$
- $89A = 10C + B$

Solve for the Digits The expression $10C + B$ is the value of the two-digit number CB. So the equation is **89A = CB**.

Since A, B, and C are distinct, non-zero digits, CB must be a two-digit number (less than 100).

If **A = 1**, then $89 \times 1 = 89$. This means CB = 89, so **C = 8** and **B = 9**.

If **A = 2**, then $89 \times 2 = 178$. This is a three-digit number, which is not possible for CB. The only solution is A=1, B=9, C=8.

Calculate the Final Sum

- $A + B + C = 1 + 9 + 8 = 18$ 🧠

Problem 12:

There are **125** such integers.

To find the number of integers that meet all the conditions, we can determine the number of choices for each of the four digits.

Analyze the Conditions

The number must be a 4-digit integer.

It must contain **only odd digits**. The set of odd digits is {1, 3, 5, 7, 9}.

It must be **divisible by 5**. A number is divisible by 5 if its last digit is 0 or 5.

Determine the Choices for Each Digit

Ones digit: Because the number must be divisible by 5 and can only contain odd digits, the last digit **must be 5**. (1 choice)

Thousands digit: This can be any of the 5 odd digits {1, 3, 5, 7, 9}. (5 choices)

Hundreds digit: This can be any of the 5 odd digits {1, 3, 5, 7, 9}. (5 choices)

Tens digit: This can be any of the 5 odd digits {1, 3, 5, 7, 9}. (5 choices)

Calculate the Total To find the total number of possible integers, we multiply the number of choices for each digit.

5 (thousands) × 5 (hundreds) × 5 (tens) × 1 (ones) = 125 🔢

Problem 13:

The value of PA + PB + PC + PD is **$6 + 6\sqrt{3}$.**

Understand the Minimization Principle The sum of the distances from a point **P** to the four vertices of a convex quadrilateral (PA + PB + PC + PD) is minimized when **P** is located at the intersection of the diagonals. The minimum value of this sum is equal to the sum of the lengths of the diagonals (AC + BD).

Analyze the Rhombus and Find its Diagonals We are given a rhombus with a side length of 6 and an angle of 60°.

Short Diagonal (AC): A rhombus with a 60° angle can be split into two identical **equilateral triangles**. In our case, triangle ABC has sides AB=6, BC=6 and the angle between them is 60°, making it equilateral. Therefore, the short diagonal **AC** must also have a length of **6**.

Long Diagonal (BD): The diagonals of a rhombus bisect each other at right angles. This creates four right-angled triangles. Let's look at the triangle formed by sides AB, AO, and BO, where O is the intersection of the diagonals.

Hypotenuse AB = 6.

Leg AO is half of AC, so AO = 6 / 2 = 3.

We can find the other leg BO (half of the long diagonal) using the Pythagorean theorem: $BO^2 + AO^2 = AB^2$ $BO^2 + 3^2 = 6^2$ $BO^2 + 9 = 36$ $BO^2 = 27$ $BO = \sqrt{27} = 3\sqrt{3}$ The full length of the long diagonal **BD** is $2 \times 3\sqrt{3} = 6\sqrt{3}$.

Calculate the Minimum Sum The minimum sum is the sum of the lengths of the two diagonals.

- Sum = AC + BD = $6 + 6\sqrt{3}$ 💎

Problem 14:

Richard has bowled **30** league games thus far.

This problem can be solved by balancing the points above and below the new average.

Calculate the Surplus Points from Today's Games Richard's new average is **182**. Let's see how many points *above* this new average he scored in his three games today.

- Game 1: 193 - 182 = +11 points
- Game 2: 207 - 182 = +25 points
- Game 3: 200 - 182 = +18 points
- Total surplus = 11 + 25 + 18 = 54 points.

Find the Number of Previous Games Let **n** be the number of games Richard bowled before today. His old average was **180**. Each of these **n** games is 182 - 180 = 2 points *below* his new average. The total point "deficit" from these old games must balance the 54-point surplus from his new games.

- 2 (points below average) × n (games) = 54 (total surplus)
- 2n = 54
- n = 27 So, Richard had bowled **27** games before today.

Calculate the Total Number of Games The question asks for the total number of games bowled so far.

27 (previous games) + 3 (today's games) = 30 games 🎳

Problem 15:

The number **335** cannot appear in Alice’s list.

Analyze the Rules of the Sequence The sequence starts with 1. Every number after that is generated from the previous number by one of two rules:

Rule 1: Add 6.

Rule 2: Multiply by 4.

Find a Common Property Let's examine the properties of the numbers in the sequence, specifically their remainders when divided by 3.

> The starting number is **1**, which has a remainder of 1 when divided by 3 (1 mod 3 = 1).
>
> Let's see what happens when we apply the rules to a number that has a remainder of 1 when divided by 3:

Rule 1 (Add 6): Adding 6 to a number doesn't change its remainder when divided by 3 (since 6 is a multiple of 3). So, the result will still have a remainder of 1.

Rule 2 (Multiply by 4): Multiplying a number with a remainder of 1 by 4 gives a result with a remainder of $1 \times 4 = 4$, and 4 has a remainder of 1 when divided by 3. This means that **every number** in Alice's list must have a remainder of 1 when divided by 3.

Test the Options Now we can check which of the given numbers does *not* have a remainder of 1 when divided by 3. A quick way to do this is to sum the digits of each number.

- **A) 109:** $1 + 0 + 9 = 10$. ($10 \div 3$ has a remainder of 1)
- **B) 151:** $1 + 5 + 1 = 7$. ($7 \div 3$ has a remainder of 1)
- **C) 244:** $2 + 4 + 4 = 10$. ($10 \div 3$ has a remainder of 1)
- **D) 335:** $3 + 3 + 5 = 11$. ($11 \div 3$ has a remainder of 2)
- **E) 412:** $4 + 1 + 2 = 7$. ($7 \div 3$ has a remainder of 1)

The only number that does not have a remainder of 1 when divided by 3 is **335**. 💡