First Round 2022-2023

Solution:

Problem 1:

The correct option is **E) None of preceding**.

Simplify the Expression for X The key to finding the relationship is to rewrite the improper fractions in the expression for X.

$$2019/2018 = (2018 + 1) / 2018 = 1 + 1/2018$$

$$2018/2019 = (2019 - 1) / 2019 = 1 - 1/2019$$

$$2019/2020 = (2020 - 1) / 2020 = 1 - 1/2020$$

Now, substitute these back into the expression for X:

- $X = (1 + 1/2018) + (1 - 1/2019) + (1 - 1/2020)$
- $X = 3 + 1/2018 - 1/2019 - 1/2020$

Combine X and Y Let's add the expression for Y to our simplified expression for X.

- $X + Y = (3 + 1/2018 - 1/2019 - 1/2020) + (1/2020 + 1/2019 - 1/2018)$
- Notice that all the fractional parts will cancel each other out:
  - 1/2018 cancels with -1/2018
  - -1/2019 cancels with +1/2019
  - -1/2020 cancels with +1/2020
- This leaves us with: $X + Y = 3$

Check the Options The correct relationship is $X + Y = 3$. Let's check the given options:

- A) $X - Y = 4$
- B) $X + Y = 1$
- C) $Y = X + 4$ (or $Y - X = 4$)
- D) $X = 4 - Y$ (or $X + Y = 4$) Since our calculated result $X + Y = 3$ does not match any of the options from A to D, the correct answer is **E) None of preceding**. 💡

Problem 2:

The value of $a + b + c$ is **19**.

Analyze the Equations We are given two equations and know that **a, b,** and **c** are distinct prime numbers.

- $a(c - b) = 18$
- $b(c - a) = 40$

From the first equation, **a** must be a prime factor of 18. The prime factors of 18 are 2 and 3. So, **a** can be **2** or **3**. From the second equation, **b** must be a prime factor of 40. The prime factors of 40 are 2 and 5. So, **b** can be **2** or **5**.

Test the Possibilities Since **a** and **b** must be *distinct* prime numbers, we can test the possible pairs.

Case 1: a = 2 If $a = 2$, then b cannot be 2. So, b must be 5. Let's use the first equation: $2(c - 5) = 18 \rightarrow c - 5 = 9 \rightarrow c = 14$. Since 14 is not a prime number, this case is incorrect.

Case 2: a = 3 If $a = 3$, then b cannot be 3. So, b must be 2 or 5.

If **b = 2**: $3(c - 2) = 18 \rightarrow c - 2 = 6 \rightarrow c = 8$. (8 is not prime).

If **b = 5**: $3(c - 5) = 18 \rightarrow c - 5 = 6 \rightarrow$ **c = 11**. (11 is a prime number and distinct from a and b).

Verify the Solution The only possibility that works is **a = 3, b = 5, c = 11**. Let's check these values with the second equation:

- $b(c - a) = 5(11 - 3) = 5(8) = 40$. This is correct.

Calculate the Final Sum

- $a + b + c = 3 + 5 + 11 = 19$ 🧠

Problem 3:

The value of $m\angle AED$ is **75°**.

Find the Sum of Angles A and D The sum of the interior angles in any quadrilateral is **360°**. We can use this to find the sum of the two unknown angles, $\angle DAB$ and $\angle ADC$.

- $\angle DAB + \angle ADC + \angle B + \angle C = 360°$
- $\angle DAB + \angle ADC + 70° + 80° = 360°$
- $\angle DAB + \angle ADC = 360° - 150° = 210°$

Use the Angle Bisector Property We are told that AE and DE are angle bisectors. This means:

∠DAE = (1/2)∠DAB

∠ADE = (1/2)∠ADC We can find the sum of these two smaller angles:

∠DAE + ∠ADE = (1/2)∠DAB + (1/2)∠ADC

∠DAE + ∠ADE = (1/2) * (∠DAB + ∠ADC)

∠DAE + ∠ADE = (1/2) * 210° = 105°

Find ∠AED The angles in the triangle **ADE** must sum to **180°**. We know the sum of the other two angles is 105°.

∠AED + ∠DAE + ∠ADE = 180°

∠AED + 105° = 180°

∠AED = 180° - 105° = 75° 📐

Problem 4:

The correct answer is **C) M – 5^{19} · 6**.

Analyze the Expressions Let's call the second expression S.

- $M = 5^2 + 5^3 + 5^4 + \ldots + 5^{18} + 5^{19} + 5^{20}$
- $S = 5^2 + 5^3 + 5^4 + \ldots + 5^{18}$ By comparing the two, we can see that **M** is the same as **S**, but with two extra terms at the end.

Form a Relationship We can write an equation connecting M and S:

- $M = S + 5^{19} + 5^{20}$ The question asks for the value of S, so we can rearrange this equation:
- $S = M - (5^{19} + 5^{20})$

Simplify the Expression To match the answer choices, we need to simplify the part in the parenthesis by factoring out the common term, 5^{19}.

- $5^{19} + 5^{20} = 5^{19}\ (1 + 5)$
- $= 5^{19}\ (6)$ Substituting this back into our equation for S gives:
- $S = M - 5^{19} \cdot 6$ 💡

Problem 5:

Let the three sides of the isosceles triangle be represented by **integers x, x, and y.**

The perimeter of the triangle is given as 200 units. So, the sum of the lengths of the sides is: $2x+y=200$

According to the triangle inequality theorem, the sum of the lengths of any two sides of a triangle must be greater than the length of the third side. Applying this theorem to our isosceles triangle, we get two main conditions:

1. $x+x>y \Rightarrow 2x>y$
2. $x+y>x \Rightarrow y>0$

From the perimeter equation, we can express y in terms of x: $y=200-2x$

Since y must be a positive integer, we have: $200-2x>0$ $200>2x$ $100>x$

Now, we substitute the expression for y into the first inequality from the triangle inequality theorem:

$2x>200-2x$,

$4x>200$,

$x>50$

So, the possible integer values for the side x must be greater than 50 and less than 100. This means the possible values for x are

$51,52,53,\ldots,99$.

To find the number of possible values for x, we can subtract the smallest value from the largest value and add 1:

Number of values = $99-51+1=49$

Each of these 49 integer values for x will create a distinct isosceles triangle. We should also consider if an equilateral triangle (where $x=y$) is possible. If $x=y$, the perimeter equation becomes $3x=200$. This gives $x=200/3$, which is not an integer. Therefore, none of the triangles are equilateral.

The correct answer is **49.**

Problem 6:

There are **4** integers for which the expression is prime.

Factor the Expression First, let's factor the quadratic expression inside the absolute value bars:

- $n^2 - 6n + 5 = (n - 1)(n - 5)$ So, the expression we are working with is $|(n - 1)(n - 5)|$.

Use the Definition of a Prime Number A prime number is a positive integer greater than 1 whose only positive integer factors are 1 and itself. Since **n** is an integer, n - 1 and n - 5 are also integers. For their product to be a prime number, the absolute value of one of the factors must be **1**. This gives us two cases to check.

Solve the Two Cases

- **Case A: $|n - 1| = 1$** This means $n - 1 = 1$ or $n - 1 = -1$.

If $n - 1 = 1$, then **$n = 2$**. The expression becomes $|(2-1)(2-5)| = |1 \times -3| = 3$, which is prime.

If $n - 1 = -1$, then **$n = 0$**. The expression becomes $|(0-1)(0-5)| = |-1 \times -5| = 5$, which is prime.

- **Case B: $|n - 5| = 1$** This means $n - 5 = 1$ or $n - 5 = -1$.

If $n - 5 = 1$, then **$n = 6$**. The expression becomes $|(6-1)(6-5)| = |5 \times 1| = 5$, which is prime.

If $n - 5 = -1$, then **$n = 4$**. The expression becomes $|(4-1)(4-5)| = |3 \times -1| = 3$, which is prime.

Count the Solutions The integer values for **n** that make the expression prime are **0, 2, 4, and 6**. There are a total of **4** such integers. 🧠

Problem 7:

There are **56** ways to select the integers.

This is a classic combinatorics problem that can be solved with a clever counting method often called "stars and bars" or the "gaps method."

Frame the Problem We need to choose 3 integers from the set {1, 2, ..., 10} such that there is at least one number between each of the integers we pick. Imagine the 10 numbers as 10 empty slots. Choosing 3 non-consecutive integers is like placing 3 "chosen" markers and 7 "unchosen" markers in these slots, with the rule that no two "chosen" markers can be next to each other.

Use the "Gaps" Method The easiest way to ensure the 3 chosen numbers are not consecutive is to first place the 7 *unchosen* numbers.

- _ U _ U _ U _ U _ U _ U _ U _ Placing the 7 unchosen numbers (U) creates 8 possible gaps (including the ends) where we can place our 3 chosen numbers.

Calculate the Combinations We need to choose 3 of these 8 available gaps to place our 3 numbers. The order in which we choose the gaps doesn't matter. This is a combination problem.

- Number of ways = C(8, 3)
- C(8, 3) = (8 × 7 × 6) / (3 × 2 × 1)
- C(8, 3) = 8 × 7 = 56

There are **56** ways to select three nonconsecutive integers from 1 to 10. 🧠

Problem 8:

The sum of all real solutions is **13**.

To solve an equation of the form base^exponent = 1, there are three possible cases to consider for the real solutions.

Case 1: The exponent is 0. The exponent is 2x - 6. We set it to 0 and solve for x.

- 2x - 6 = 0
- 2x = 6
- x = 3 *(We must check that the base, x - 5, is not 0 at this value. For x=3, the base is -2, so this solution is valid.)*

Case 2: The base is 1. The base is x - 5. We set it to 1 and solve for x.

- x - 5 = 1
- x = 6

Case 3: The base is -1 and the exponent is an even integer. First, we set the base x - 5 to -1.

- x - 5 = -1
- x = 4 Next, we must check if the exponent 2x - 6 is an even integer for this value of x.
- 2(4) - 6 = 8 - 6 = 2 Since 2 is an even integer, this solution is also valid.

Find the Sum The real solutions for x are **3, 4, and 6**. The sum of these solutions is:

- $3 + 4 + 6 = 13$ 🧠

Problem 9:

The number of positive divisors is **12**.

Translate the Notation The notation xx represents a two-digit number where both digits are the same. We can express this algebraically:

- $xx = 10x + x = 11x$
- $yy = 10y + y = 11y$
- $zz = 10z + z = 11z$

Rewrite and Simplify the Expression Now, let's substitute these into the expression we need to evaluate.

- $(xx)^2 + (yy)^2 + (zz)^2 = (11x)^2 + (11y)^2 + (11z)^2$
- $= 121x^2 + 121y^2 + 121z^2$ We can factor out the 121:
- $= 121(x^2 + y^2 + z^2)$

Calculate the Value The problem gives us the value of $x^2 + y^2 + z^2 = 74$. We can now find the numerical value of the expression.

- $\text{Value} = 121 \times 74$

Find the Number of Divisors To find the number of divisors, we first need the prime factorization of the value.

- $121 = 11^2$
- $74 = 2 \times 37$ So, the prime factorization of the number is $\mathbf{2^1 \times 11^2 \times 37^1}$. The number of divisors is found by adding 1 to each exponent and multiplying the results.
- $\text{Number of divisors} = (1 + 1) \times (2 + 1) \times (1 + 1)$
- $= 2 \times 3 \times 2 = 12$ 🔢

Problem 10:

The area of the right triangle ABC is **28**.

Use the Pythagorean Theorem For any right triangle, the sides are related by the Pythagorean theorem: $\mathbf{a^2 + b^2 = c^2}$. We can use this to simplify the second given equation.

- We are given: $a^2 + b^2 + c^2 = 288$
- Substitute c^2 for $a^2 + b^2$: $c^2 + c^2 = 288$

- $2c^2 = 288$
- $c^2 = 144$
- $c = 12$

Find the Sum of the Legs (a + b) Now that we know the hypotenuse $c = 12$, we can use the first equation to find the sum of the other two sides.

- We are given: $a + b + c = 28$
- $a + b + 12 = 28$
- $a + b = 16$

Find the Product of the Legs (ab) The area of the triangle is $(1/2)ab$, so we need to find the product ab. We can find this using the algebraic identity $(a + b)^2 = a^2 + b^2 + 2ab$.

- We know $a + b = 16$, so $(a + b)^2 = 16^2 = 256$.
- We know $a^2 + b^2 = c^2 = 144$.
- Substitute these values into the identity: $256 = 144 + 2ab$ $112 = 2ab$ $ab = 56$

Calculate the Area

- Area $= (1/2)ab = (1/2) \times 56 = 28$ 📐

Problem 11:

The probability is **21/32**.

Understand the Condition Maria flips six fair coins. "At least half" of them landing heads means she could get **3, 4, 5, or 6 heads**. We need to find the probability of all these outcomes combined.

Calculate the Total Possible Outcomes Each of the six coins has two possible outcomes (heads or tails).

- Total outcomes $= 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$.

Calculate the Favorable Outcomes We need to find the number of ways to get 3, 4, 5, or 6 heads. We can use the combination formula C(n, k) for this, where n is the number of coins and k is the number of heads.

- **Ways to get 3 heads:** $C(6, 3) = (6\times5\times4)/(3\times2\times1) = 20$
- **Ways to get 4 heads:** $C(6, 4) = (6\times5)/(2\times1) = 15$
- **Ways to get 5 heads:** $C(6, 5) = 6$

- **Ways to get 6 heads:** C(6, 6) = 1 The total number of favorable outcomes is the sum of these possibilities:
- 20 + 15 + 6 + 1 = 42

Find the Probability The probability is the ratio of favorable outcomes to the total number of outcomes.

- Probability = 42 / 64
- Simplifying this fraction by dividing both numbers by 2 gives **21/32**. 🪙

Problem 12:

The correct answer is **192**.

Simplify the Ratio The problem states the integer side lengths are in the ratio 1 : 3/2 : 2. To work with whole numbers, we can multiply all parts of the ratio by 2 to clear the fraction.

- (1 × 2) : (3/2 × 2) : (2 × 2)
- 2 : 3 : 4 So, the side lengths are in the ratio **2 : 3 : 4**.

Set Up the Volume Expression We can represent the three integer side lengths as **2x**, **3x**, and **4x**, where x is some positive integer. The volume of the box is the product of these lengths.

- Volume = (2x) × (3x) × (4x)
- Volume = $24x^3$

Test the Options This formula tells us that the volume must be a multiple of 24, and the result of dividing the volume by 24 must be a perfect cube (x^3). Let's check the options:

- 120 / 24 = 5 (not a perfect cube)
- 144 / 24 = 6 (not a perfect cube)
- 168 / 24 = 7 (not a perfect cube)
- **192 / 24 = 8** (which is **2^3**, a perfect cube)
- 216 / 24 = 9 (not a perfect cube)

The only option that satisfies the condition is **192**. In this case, x would be 2, and the side lengths would be 4, 6, and 8. 📦

Problem 13:

The minimum possible value of x + y is **40**.

Set Up the Sums We use the formula for the sum of the first n terms of an arithmetic sequence: $S_n = n/2 * (2a + (n-1)d)$, where a is the first term and d is the common difference.

For the first sequence: a=1, d=x, n=20. $Sum_1 = 20/2 * (2(1) + (20-1)x) = 10(2 + 19x) = 20 + 190x$

For the second sequence: a=10, d=y, n=10. $Sum_2 = 10/2 * (2(10) + (10-1)y) = 5(20 + 9y) = 100 + 45y$

Form an Equation The problem states that the two sums are equal.

- 20 + 190x = 100 + 45y
- 190x - 45y = 80 We can simplify this equation by dividing all terms by 5:
- 38x - 9y = 16

Find the Smallest Positive Integer Solution We need to find the pair of positive integers (x, y) that satisfies 38x - 9y = 16 and has the smallest possible sum x + y. Let's rearrange the equation to solve for y:

9y = 38x - 16 Since x and y must be integers, 38x - 16 must be divisible by 9. We can test positive integer values for x starting from 1 until we find one that works.

If x=1, 38(1)-16 = 22 (not divisible by 9)

If x=2, 38(2)-16 = 60 (not divisible by 9)

...

If **x=8**, 38(8)-16 = 304 - 16 = 288. 288 is divisible by 9 (288 / 9 = 32). This gives us our first (and smallest) positive integer solution: **x = 8** and **y = 32**.

Calculate the Minimum Sum The sum for this pair is:

- x + y = 8 + 32 = 40 🧠

Problem 14:
The sum of the digits is **630**.

Rewrite the Multiplication The number **N** is formed by 105 ones. Instead of multiplying by 105 directly, we can rewrite the multiplication as (100 + 5) × N, which is equal to 100N + 5N.

- **100N:** This is the number N with two zeros added at the end. It's 105 ones followed by 00.
- **5N:** This is 5 × 111...1 (105 times), which gives 555...5 (105 fives).

Add the Two Parts Now, let's add these two numbers together. We can see the pattern by looking at the last few digits:

```
  ...1111100
+   ...555555
-----------
...666655
```

- The last two digits are 00 + 55 = 55.
- The digits before that are 1 + 5 = 6. This continues for many places.

Determine the Final Number's Digits Based on the addition, the resulting number has a clear pattern:

- It starts with two 1s.
- Followed by 105 - 2 = 103 sixes.
- It ends with two **5**s. The number looks like: 11666...6655 (with 103 sixes).

Calculate the Sum of the Digits Finally, we add up all the digits of this new number.

- Sum of the 1s: 1 + 1 = 2
- Sum of the 6s: 103 × 6 = 618
- Sum of the 5s: 5 + 5 = 10
- **Total Sum:** 2 + 618 + 10 = 630 🧠

Problem 15:

The eighth player scored **6** points.

Find the Total Points Awarded The key to this problem is that in every game, a total of exactly **1 point** is awarded, whether it's a win (1 + 0 = 1) or a tie (0.5 + 0.5 = 1). Therefore, the sum of all the players' final scores is equal to the total number of games played.

Calculate the Total Number of Games In a round-robin tournament with 8 players, each player plays every other player once. The total number of games is the number of ways to choose 2 players from 8, which is C(8, 2).

- Total Games = (8 × 7) / (2 × 1) = 28 games.
- This means the total score for all 8 players combined must be **28 points**.

Sum the Known Scores Next, we add up the scores of the seven players that are given.

- 0 + 1 + 2.5 + 3 + 3.5 + 5 + 7 = 22 points.

Find the Eighth Player's Score The score of the eighth player is the difference between the total points and the sum of the other players' scores.

- 28 (total points) - 22 (sum of 7 players) = 6 points. ♟