ARCHIMEDES MATH COMPETITION

First Round 2022-2023

Solution:

Problem 1:

There are **3** possible integer values for the length of the third side.

Understand the Triangle Inequality Theorem For any valid triangle, the length of any side must be less than the sum of the other two sides and greater than their difference. If we call the sides a, b, and c:

$(a - b) < c < (a + b)$

Calculate the Range for the Third Side Let's use the given side lengths, $a = 6.3$ and $b = 1.7$, to find the possible range for the third side, c.

Difference: $6.3 - 1.7 = 4.6$

Sum: $6.3 + 1.7 = 8.0$ So, the length of the third side c must be between 4.6 and 8.0:

$4.6 < c < 8.0$

Count the Possible Integer Values The question asks for the number of integer values possible for c. The integers that are greater than 4.6 and less than 8.0 are:

5, 6, and 7 Counting these, we find there are **3** possible values.

Problem 2:

The value of c is 8.

Use the Property of the Center Square

In a 3x3 magic square, the sum of any row, column, or diagonal (the "magic sum") is always 3 times the value of the center cell.

We are given that the magic sum is 42.

The center cell is $2x + 2$.

This gives us the equation: $3 * (2x + 2) = 42$

Solve for x

Now we can solve this simple equation to find the value of x.

$2x + 2 = 42 / 3$

$2x + 2 = 14$

$2x = 12$

$x = 6$

Find the Value of c

The sum of the numbers on the main diagonal (from top-left to bottom-right) must also be 42.

$c + (2x + 2) + 20 = 42$

Substitute the value $x=6$ into this equation:

$c + (2*6 + 2) + 20 = 42$

$c + (12 + 2) + 20 = 42$

$c + 14 + 20 = 42$

$c + 34 = 42$

$c = 42 - 34 = 8$ 🔢

Problem 3:

The probability is **8/15**.

Understand the Condition For the product of two numbers to be a **positive number**, both numbers must have the same sign. This means we need to select either **two positive numbers** or **two negative numbers**.

Analyze the Set The given set is $A = \{-7, -6, -5, -4, -3, -2, -1, 1, 2, 3\}$.

- Total numbers in the set: **10**
- Number of positive numbers: **3** ($\{1, 2, 3\}$)
- Number of negative numbers: **7** ($\{-7, -6, \ldots, -1\}$)

Calculate the Total Possible Outcomes First, we find the total number of ways to choose any two numbers from the set of 10.

- Total ways = $C(10, 2) = (10 \times 9) / (2 \times 1) = 45$

Calculate the Favorable Outcomes Next, we find the number of ways to choose two numbers with the same sign.

- **Ways to choose 2 positive numbers** from the 3 available: C(3, 2) = 3
- **Ways to choose 2 negative numbers** from the 7 available: C(7, 2) = (7 × 6) / (2 × 1) = 21
- Total favorable outcomes = 3 + 21 = 24

Find the Probability The probability is the ratio of favorable outcomes to total outcomes.

- Probability = 24 / 45
- Simplifying the fraction by dividing both numbers by 3 gives **8/15**.

Problem 4:

The sum of all possible x values is **4**.

To find the possible values for x, we can apply the divisibility rules one by one to narrow down the options for the digits x and y.

Divisibility by 5 A number is divisible by 5 if its last digit is 0 or 5.

This means **y** must be either **0** or **5**.

Divisibility by 4 A number is divisible by 4 if the number formed by its last two digits is divisible by 4. The last two digits form the number xy.

Case 1 (y=5): The number would be x5. A number ending in 5 is always odd and can never be divisible by 4. So, this case is impossible.

Case 2 (y=0): The number is x0. For x0 to be divisible by 4, **x** must be an even digit: {0, 2, 4, 6, 8}.

From these two rules, we know **y = 0** and **x** must be one of **{0, 2, 4, 6, 8}**.

Divisibility by 3 A number is divisible by 3 if the sum of its digits is a multiple of 3.

The sum of the digits is 5 + 3 + x + y.

Since we know y = 0, the sum is 8 + x. Now, we test our possible values for **x** to see which one makes 8 + x divisible by 3.

- If x = 0, 8 + 0 = 8 (Not divisible by 3)
- If x = 2, 8 + 2 = 10 (Not divisible by 3)
- If x = 4, 8 + 4 = 12 (Divisible by 3) → **This is a valid solution.**

- If x = 6, 8 + 6 = 14 (Not divisible by 3)
- If x = 8, 8 + 8 = 16 (Not divisible by 3)

Find the Sum The only possible value for x is **4**. Since there is only one value, the sum of all possible x values is simply 4. 🔢

Problem 5:

The perimeter of the triangle is **19 cm**.

Set Up the Equations Let the side lengths of the triangle be **PQ**, **QR**, and **PR**. We are given the following:

1. PQ + QR = 10
2. QR + PR = 12
3. PR + PQ = 16

Combine the Equations The quickest way to find the perimeter (PQ + QR + PR) is to add all three equations together.

(PQ + QR) + (QR + PR) + (PR + PQ) = 10 + 12 + 16

When we group the terms, we see that each side length appears twice: 2(PQ) + 2(QR) + 2(PR) = 38

Find the Perimeter We can factor out the 2 from the left side.

2 * (PQ + QR + PR) = 38 Since PQ + QR + PR is the perimeter, this means twice the perimeter is 38. To find the perimeter, simply divide by 2.

Perimeter = 38 / 2 = 19 📐

Problem 6:

Let's analyze the updated image carefully.

We see diagrams consisting of squares arranged in an n×nn \times nn×n grid for each diagram number nnn, where some squares are shaded and some are not.

Observing the Pattern:

Diagram 1:

- 1×1=11 \times 1 = 11×1=1 square
- All shaded → Shaded: 1, Unshaded: 0
- Difference: 1−0=11 - 0 = 11−0=1

Diagram 2:

- 2×2=42 \times 2 = 42×2=4 squares
- Shaded: 3, Unshaded: 1
- Difference: 3−1=23 - 1 = 23−1=2

Diagram 3:

- 3×3=93 \times 3 = 93×3=9 squares
- Shaded: 5, Unshaded: 4
- Difference: 5−4=15 - 4 = 15−4=1

Diagram 4:

- 4×4=164 \times 4 = 164×4=16 squares
- Shaded: 9, Unshaded: 7
- Difference: 9−7=29 - 7 = 29−7=2

Pattern:

The **difference** between shaded and unshaded squares alternates:

- Diagram 1: difference = 1
- Diagram 2: difference = 2
- Diagram 3: difference = 1
- Diagram 4: difference = 2

So, the **odd-numbered diagrams** have a difference of 1 and the **even-numbered diagrams** have a difference of 2.

For the 50th Diagram:

Since 50 is **even**, the difference will be **2 × 25 = 50**

Problem 7:

The number 100n has **54** positive divisors.

Analyze the Properties of n

We are told that the number **n** ends in 99. Any number ending in 99 is not divisible by 2 (since it's odd) and not divisible by 5 (since it doesn't end in 0 or 5).

The prime factors of 100 are 2 and 5. This means that **n** and **100** share no common prime factors; they are **coprime**.

Use the Divisor Function Property A key property of the number of divisors function, d(x), is that it's multiplicative for coprime numbers. This means if a and b are coprime, then $d(a \times b) = d(a) \times d(b)$.
Since n and 100 are coprime, we can say: $d(100n) = d(100) \times d(n)$

Calculate the Parts

We are given that **n** has exactly six positive divisors, so $d(n) = 6$.
We need to find the number of divisors of 100. First, find its prime factorization: $100 = 10^2 = (2 \times 5)^2 = 2^2 \times 5^2$.
The number of divisors is $(2+1) \times (2+1) = 3 \times 3 = 9$. So, $d(100) = 9$.

Find the Final Answer Now, we can find the number of divisors of 100n.
$d(100n) = d(100) \times d(n)$
$d(100n) = 9 \times 6 = 54$ 🧠

Problem 8:

There are **10** integers that satisfy the double inequality.
To find the number of integers, we can solve the double inequality for **n**. It's easiest to split it into two separate parts.

Solve the First Inequality

$5/19 < 6/n$
$5n < 6 \times 19$
$5n < 114$
$n < 114 / 5$
$n < 22.8$

Solve the Second Inequality

$6/n < 1/2$
$6 \times 2 < n$
$12 < n$

Combine the Results and Count We now know that **n** must be greater than 12 and less than 22.8.

$12 < n < 22.8$ The integers that satisfy this condition are {13, 14, 15, 16, 17, 18, 19, 20, 21, 22}. Counting these numbers, we find there are a total of **10** integers. 🔢

Problem 9:

The value of a + b + c + d is **8**.

This type of problem, involving a continued fraction, can be solved by repeatedly converting the improper fraction into a mixed number.

Find the value of a The expression a + ... shows that **a** is the integer part of the fraction 23/7.

23 / 7 = 3 with a remainder of 2.

So, 23 / 7 = 3 + 2/7.

By comparing this to the original equation, we find that **a = 3**.

Now we know that 1 / (b + 1 / (c + 1/d)) = 2/7.

Find the value of b We can take the reciprocal of both sides of the remaining equation:

b + 1 / (c + 1/d) = 7/2 Now, **b** is the integer part of 7/2.

7 / 2 = 3 with a remainder of 1.

So, 7 / 2 = 3 + 1/2.

This tells us that **b = 3**. Now we know that 1 / (c + 1/d) = 1/2.

Find the values of c and d Again, we take the reciprocal of both sides:

c + 1/d = 2 Since **c** and **d** must both be positive integers, the only way for 1/d to be an integer is if **d = 1**.

If d = 1, the equation becomes c + 1 = 2, which means **c = 1**.

Calculate the Final Sum We have found all four integers:

a = 3

b = 3

c = 1

d = 1 The sum is 3 + 3 + 1 + 1 = 8. 🧠

Problem 10:

The correct piece is **A**. (Note: Option D is also a valid possibility).

Understand the Grid Rules In the 5-column table, there are two simple rules for any number n:

- The number to its right is n + 1.

- The number directly below it is n + 5.

Test the Options We need to find which 2x2 piece follows these rules.

- **Option A:** Shows 43 in the top-left and 48 in the bottom-left.

According to the rule, the number below 43 should be 43 + 5 = 48. This matches. **This is a possible answer.**

- **Option B:** Shows 58 in the top-right and 52 in the bottom-left.

The number to the left of 58 would be 57. The number below 57 should be 57 + 5 = 62. The piece shows 52, so this is impossible.

- **Option C:** Shows 69 in the top-right and 72 in the bottom-left.

The number to the left of 69 would be 68. The number below 68 should be 68 + 5 = 73. The piece shows 72, so this is impossible.

- **Option D:** Shows 81 in the top-left and 86 in the bottom-left.

The number below 81 should be 81 + 5 = 86. This matches. **This is also a possible answer.**

- **Option E:** Shows 90 in the top-left and 94 in the bottom-right.

The number to the right of 90 would be 91. The number below 91 should be 91 + 5 = 96. The piece shows 94, so this is impossible.

Conclusion Both options **A** and **D** perfectly follow the rules of the grid. However, in a standard test format, you would typically select the first valid option you find. 🧩

Problem 11:

The correct answer is **192**.

Simplify the Ratio The problem states the integer side lengths are in the ratio 1 : 3/2 : 2. To work with integers, we can multiply all parts of the ratio by 2 to clear the fraction.

- (1 × 2) : (3/2 × 2) : (2 × 2)
- 2 : 3 : 4 So, the side lengths are in the ratio **2 : 3 : 4**.

Set Up the Volume Expression We can represent the three integer side lengths as **2x**, **3x**, and **4x**, where x is some positive integer. The volume of the box is the product of these lengths.

- Volume = (2x) × (3x) × (4x)

- Volume = $24x^3$

Test the Options This formula tells us that the volume must be a multiple of **24**. Now we can check which of the options is divisible by 24.

- 136 ÷ 24 = 5.66...
- 148 ÷ 24 = 6.16...
- 160 ÷ 24 = 6.66...
- **192 ÷ 24 = 8**

204 ÷ 24 = 8.5 The only option that is a multiple of 24 is **192**.

Problem 12:

The correct answer is **A) 2-A**.

The key to this problem is to rewrite the fractions by separating them into an integer part and a fractional part.

Analyze the Expression for A Let's start with the given equation for **A**.

- A = 21/19 + 11/29
- We can rewrite the first term: 21/19 = (19 + 2) / 19 = 1 + 2/19.
- So, A = 1 + 2/19 + 11/29.

Analyze the Expression to Find Now let's look at the expression we need to find.

- 18/29 - 2/19
- We can rewrite the first term: 18/29 = (29 - 11) / 29 = 1 - 11/29.
- So, the expression becomes (1 - 11/29) - 2/19.

Find the Relationship Let's rearrange the two expressions to see how they relate.

- From our analysis of A, we can write: A - 1 = 2/19 + 11/29.
- From our analysis of the second expression, we can write: 1 - (11/29 + 2/19). Notice that the part in the parenthesis, (2/19 + 11/29), is the same in both. We can substitute A - 1 into the second expression:
- 1 - (A - 1)
- 1 - A + 1 = 2 - A

The expression 18/29 - 2/19 is equal to **2 - A**. 💡

Problem 13:

The number of pages in the book is **63**.

Set Up the Problem Let **n** be the number of pages in the book. The sum of the first **n** page numbers is given by the formula n(n+1)/2. The problem gives us two conditions:

- The sum of the pages is less than 2020: $n(n+1)/2 < 2020$
- If there was one more page, the sum would be more than 2020: $(n+1)(n+2)/2 > 2020$

Estimate the Number of Pages (n) We can find an approximate value for **n** by setting the sum equal to 2020.

- $n(n+1)/2 = 2020$
- $n(n+1) = 4040$ Since n and n+1 are very close, n^2 is approximately 4040. We can estimate n by taking the square root of 4040.
- $\sqrt{3600} = 60$
- $\sqrt{4900} = 70$ Let's try a number in between, like 63. $63 \times 64 \approx 4032$. So, **n** should be very close to 63.

Test the Values Let's test **n = 63** to see if it meets the conditions.

- **Sum for 63 pages:** $(63 \times 64) / 2 = 2016$. Is $2016 < 2020$? Yes, the first condition is met.
- **Sum for 64 pages (n+1):** $(64 \times 65) / 2 = 2080$. Is $2080 > 2020$? Yes, the second condition is met.

Since both conditions are satisfied for **n = 63**, this is the correct number of pages. 📖

Problem 14:

The value of A + B is **14**.

Find the Value of A (The Sum) We need to find the sum of all integers from -13 to 14.

- Sum = (-13) + (-12) + ... + 0 + ... + 12 + 13 + 14 In this sum, every negative number will cancel out its positive counterpart (e.g., -13 + 13 = 0, -12 + 12 = 0, and so on).
- Sum = (-13 + 13) + (-12 + 12) + ... + 0 + 14
- Sum = 0 + 0 + ... + 0 + 14 The only number left is 14. So, **A = 14**.

Find the Value of B (The Product) We need to find the product of all integers from -13 to 14.

- Product = (-13) × (-12) × ... × (-1) × 0 × 1 × ... × 13 × 14 Since the number **0** is part of this set, the entire product will be 0.
- So, **B = 0**.

Calculate A + B

- A + B = 14 + 0 = 14 🧠

Problem 15:

There are **12** such positive values of n.

To solve this, we need to find the range of possible values for **n** based on each condition and then find the overlap between those ranges.

Analyze the First Condition: n/3 is a three-digit integer

- A three-digit integer is any whole number from 100 to 999.
- $100 \leq n/3 \leq 999$
- Multiplying by 3, we get the range for n: $300 \leq n \leq 2997$.
- This condition also implies that **n must be a multiple of 3**.

Analyze the Second Condition: 3n is a three-digit integer

- $100 \leq 3n \leq 999$
- Dividing by 3, we get another range for n: $33.33... \leq n \leq 333$.
- Since n must be an integer, this range is effectively $34 \leq n \leq 333$.

Combine the Conditions and Count We need to find the numbers **n** that satisfy all conditions simultaneously:

- $300 \leq n \leq 2997$
- $34 \leq n \leq 333$

- n is a multiple of 3. The intersection of the two ranges is $300 \leq n \leq 333$. Now we just need to count how many multiples of 3 are in this range.
- The first multiple of 3 is **300** (3×100).
- The last multiple of 3 is **333** (3×111). To find the total count, we can count the multipliers from 100 to 111, inclusive:
- $111 - 100 + 1 = 12$

There are **12** possible values for n. 🔢