Solution:

Problem 1:

The correct option is **E) None of the preceding**.

Break Down the Logarithm First, we can use the properties of logarithms to break down $\log_2(60)$. We start by finding the prime factorization of 60.

$60 = 4 \times 15 = 2^2 \times 3 \times 5$ Now, apply the logarithm rules:

$\log_2(60) = \log_2(2^2 \times 3 \times 5)$

$= \log_2(2^2) + \log_2(3) + \log_2(5) = 2 + \log_2(3) + \log_2(5)$

Relate the Terms to a and b We are given $\log_2 3 = a$. The expression is now $2 + a + \log_2(5)$. We need to find $\log_2(5)$ in terms of a and b. We can do this using the **change of base formula**: $\log_x(y) = \log_k(y) / \log_k(x)$.

Let's use the two given equations: $\log_2 3 = a$ and $\log_3 5 = b$.

$\log_2 5 = \log_2 3 \times \log_3 5$ *(This is a useful identity derived from the change of base rule)*

$\log_2 5 = a \times b$

Find the Final Expression Now we can substitute everything back into our expression for $\log_2(60)$.

$\log_2(60) = 2 + \log_2(3) + \log_2(5)$

$\log_2(60) = 2 + a + ab$

The final expression is **ab + a + 2**. Since this does not match any of the options from A to D, the correct answer is **E) None of the preceding**. 💡

Problem 2:

There are **9** integer values for n.

For n^(18/n) to be an integer, there are two main possibilities: either the exponent 18/n is a positive integer, or n is a perfect power that can cancel the denominator of the fractional exponent.

Case 1: The exponent 18/n is an integer. For 18/n to be an integer, **n** must be a divisor of 18.

Positive Divisors: {1, 2, 3, 6, 9, 18}. All of these result in an integer value (e.g., $1^{18}=1$, $2^9=512$, $9^2=81$, etc.). This gives us **6** solutions.

Negative Divisors: {-1, -2, -3, -6, -9, -18}.

If n = -1, $(-1)^{-18} = 1$, which is an integer. This gives us **1** solution.

For all other negative divisors, the exponent 18/n is a negative integer, which results in a fraction (e.g., $(-2)^{-9} = 1/(-2)^9$).

So far, we have 6 + 1 = 7 solutions from this case.

Case 2: The exponent 18/n is a fraction. For the value to still be an integer, **n** must be a perfect power. Specifically, if the simplified fraction for the exponent is p/q, then **n** must be a perfect q-th power.

Test n = 4: The exponent is 18/4 = 9/2. For $4^{9/2}$ to be an integer, 4 must be a perfect square, which it is ($4=2^2$). $4^{(9/2)} = (\sqrt{4})^9 = 2^9 = 512$. This is an integer, so **n = 4** is a solution.

Test n = 8: The exponent is 18/8 = 9/4. For $8^{9/4}$ to be an integer, 8 must be a perfect 4th power, which it is not.

Test n = 27: The exponent is 18/27 = 2/3. For $27^{2/3}$ to be an integer, 27 must be a perfect cube, which it is ($27=3^3$). $27^{(2/3)} = (\sqrt[3]{27})^2 = 3^2 = 9$. This is an integer, so **n = 27** is a solution. *(No other positive or negative perfect powers up to a reasonable limit will work).*

Total Count We combine the unique solutions from both cases:

From Case 1: {1, 2, 3, 6, 9, 18, -1} (7 solutions)

From Case 2: {4, 27} (2 solutions) Total number of integer values = 7 + 2 = 9. 🧠

Problem 3:

The minimum possible value of x + y is **40**.

Set Up the Sums We use the formula for the sum of the first n terms of an arithmetic sequence: $S_n = n/2 * (2a + (n-1)d)$, where a is the first term and d is the common difference.

For the first sequence: a=1, d=x, n=20. $Sum_1 = 20/2 * (2(1) + (20-1)x) = 10(2 + 19x) = 20 + 190x$

For the second sequence: a=10, d=y, n=10. $Sum_2 = 10/2 * (2(10) + (10-1)y) = 5(20 + 9y) = 100 + 45y$

Form an Equation The problem states that the two sums are equal.

$20 + 190x = 100 + 45y$

$190x - 45y = 80$ We can simplify this equation by dividing all terms by 5:

$38x - 9y = 16$

Find the Smallest Positive Integer Solution We need to find the pair of positive integers (x, y) that satisfies $38x - 9y = 16$ and has the smallest possible sum $x + y$. Let's rearrange the equation to solve for y:

$9y = 38x - 16$ Since x and y must be integers, $38x - 16$ must be divisible by 9. We can test positive integer values for x starting from 1 until we find one that works.

If x=1, $38(1)-16 = 22$ (not divisible by 9)

If x=2, $38(2)-16 = 60$ (not divisible by 9)

...

If **x=8**, $38(8)-16 = 304 - 16 = 288$. 288 is divisible by 9 ($288 / 9 = 32$). This gives us our first (and smallest) positive integer solution: **x = 8** and **y = 32**.

Calculate the Minimum Sum The sum for this pair is:

$x + y = 8 + 32 = 40$ 🧠

Problem 4:

The remainder is **2**.

Analyze the Pattern of Remainders The key to solving this is to realize that the remainder of a large sum is the sum of the individual remainders. Let's look for a pattern when k^4 is divided by 16.

If k is an even number: We can write $k = 2m$. Then $k^4 = (2m)^4 = 16m^4$. This is always a multiple of 16, so the remainder is **0**.

If k is an odd number: We can write $k = 2m + 1$. It can be shown that the fourth power of any odd number always has a remainder of **1** when divided by 16.

Example: $1^4 = 1$, $3^4 = 81 = 16\times5 + 1$, $5^4 = 625 = 16\times39 + 1$.

Apply the Pattern to the Sum The sum $1^4 + 2^4 + 3^4 + ... + 2019^4$ divided by 16 will have a remainder equal to the sum of the individual remainders.

All the even terms (2^4, 4^4, 6^4, ...) will have a remainder of 0.

All the odd terms (1^4, 3^4, 5^4, ...) will have a remainder of 1. So, the total remainder is simply the sum of all the 1s, which is equal to the **number of odd integers** from 1 to 2019.

Count the Odd Integers The odd integers are 1, 3, 5, ..., 2019. To count them, we can use the formula (Last - First)/2 + 1.

(2019 - 1) / 2 + 1 = 2018 / 2 + 1 = 1009 + 1 = 1010 There are **1010** odd numbers from 1 to 2019.

Find the Final Remainder The sum of the remainders is 1010. To find the final answer, we need to find the remainder of 1010 when it is divided by 16.

1010 ÷ 16

16 × 63 = 1008

1010 - 1008 = 2 The final remainder is **2**. 🧠

Problem 5:

The probability is **59/125**.

Analyze the Set and Outcomes The set of numbers is {1, 2, 3, 4, 5}.

Odd numbers: {1, 3, 5} (3 choices)

Even numbers: {2, 4} (2 choices) Since the three numbers x, y, z are picked with replacement, the total number of possible outcomes is:

5 (choices for x) × 5 (choices for y) × 5 (choices for z) = 125

Condition for an Even Result For the expression xz + y to be an even number, the two terms (xz and y) must have the same parity. This gives us two possible cases:

Case 1: xz is even AND y is even.

Case 2: xz is odd AND y is odd.

Calculate the Favorable Outcomes for Each Case

Case 1: (xz = Even) + (y = Even)

Number of ways for y to be even: **2**

For xz to be even, at least one of x or z must be even. The easiest way to count this is (Total ways for xz) - (Ways for xz to be odd).

xz is only odd if both x and z are odd: 3 (odd choices) × 3 (odd choices) = 9 ways.

Total ways to choose x and z is $5 \times 5 = 25$.

So, the number of ways for xz to be even is $25 - 9 = 16$.

Total outcomes for this case = 16 (for xz) × 2 (for y) = 32.

Case 2: (xz = Odd) + (y = Odd)

Number of ways for y to be odd: **3**

Number of ways for xz to be odd (both x and z must be odd): $3 \times 3 = 9$.

Total outcomes for this case = 9 (for xz) × 3 (for y) = 27.

Calculate the Total Probability Add the outcomes from the two cases to get the total number of favorable outcomes.

$32 + 27 = 59$ The probability is the ratio of favorable outcomes to the total number of outcomes.

Probability = 59 / 125 🎲

Problem 6:

The value of $x + y$ is **7/2**.

Manipulate the Equations The key to solving this system of equations is to combine them in a way that allows us to simplify. Let's multiply the first equation by 2.

Equation 1: $2x^2 - 3y = -17/2 \rightarrow 4x^2 - 6y = -17$

Equation 2: $y^2 - 4x = 7$

Combine the Equations Now, add the new version of Equation 1 and Equation 2 together:

$$(4x^2 - 6y) + (y^2 - 4x) = -17 + 7$$

$$4x^2 - 4x + y^2 - 6y = -10$$

Complete the Square We can rearrange this new equation and complete the square for both the x and y terms.

$$(4x^2 - 4x) + (y^2 - 6y) = -10$$

$$4(x^2 - x) + (y^2 - 6y) = -10$$

$$4\left((x - 1/2)^2 - 1/4 \right) + \left((y - 3)^2 - 9 \right) = -10$$

$$4(x - 1/2)^2 - 1 + (y - 3)^2 - 9 = -10$$
$$4(x - 1/2)^2 + (y - 3)^2 - 10 = -10$$
$$4(x - 1/2)^2 + (y - 3)^2 = 0$$

Solve for x and y The only way for the sum of two squared terms to be zero is if both terms are individually zero.

$$4(x - 1/2)^2 = 0 \rightarrow x - 1/2 = 0 \rightarrow \mathbf{x = 1/2}$$
$$(y - 3)^2 = 0 \rightarrow y - 3 = 0 \rightarrow \mathbf{y = 3}$$

Calculate the Final Value The question asks for the value of $x + y$.

$$x + y = 1/2 + 3 = 3.5 \text{ or } \mathbf{7/2}$$

Problem 7:

The least number of weighings required is **2**.

This is a classic logic puzzle. The goal is to devise a strategy that *guarantees* you find 25 real coins, no matter what the outcome of the weighings are.

Here is a two-weighing solution:

Weighing 1:

Take 100 of the 101 coins and divide them into two groups of 50.

Place one group of 50 on the left side of the scale (L) and the other 50 on the right side (R). Leave the single remaining coin off the scale (O).

There are two possible outcomes:

The scale tips (e.g., L is lighter than R): This is the most informative outcome. Because the fake coins are lighter, the heavier side (R) must have fewer fake coins than the lighter side (L). With a total of 3 fake coins, this means the heavier group of 50 coins can contain at most 1 fake coin.

The scale balances: This would mean each group of 50 has exactly one fake, and the coin off the scale is also fake. This scenario is less helpful for our next step.

Let's proceed with the tipping scenario, as we must have a plan that works for any outcome.

Weighing 2:

Take the heavier group of 50 coins from the first weighing. We know for certain that this group has either 0 or 1 fake coins.

Split this group of 50 into two groups of 25.

Place one group of 25 on the left side of the scale and the other 25 on the right.
There are two possible outcomes:
The scale balances: This can only happen if there were no fake coins in the group of 50 to begin with. Therefore, both groups of 25 on the scale are **all real coins**.
The scale tips: This happens if there was one fake coin in the group of 50. The lighter side contains the single fake coin, and the heavier side consists of **25 guaranteed real coins**.
In either case, after just two weighings, we can with 100% certainty identify a group of 25 real coins. It can be shown that it's impossible to guarantee this in a single weighing. 💡

Problem 8:
The length of AK is **16**.
This geometry problem can be solved effectively using vectors.
Set Up a Vector System Let's place the parallelogram in a coordinate system with vertex **A** at the origin (0,0). We can define the sides using two vectors:
vector AB = **u**
vector AD = **v** From this, we can find the position vectors for the other vertices: B = **u**, D = **v**, and C = **u** + **v**.
Find the Position Vectors for E and F
Point E is on side BC. We are given CE = 3BE, which means BE is 1/4 of the total length of BC. vector BE = (1/4) * vector BC = (1/4) * **v**. So, the position vector for E is vector AE = vector AB + vector BE = **u** + (1/4)**v**.
Point F is the midpoint of side CD (since CF = DF). The position vector for F is the average of the vectors for C and D. vector AF = (vector AC + vector AD) / 2 = (**u** + **v** + **v**) / 2 = (1/2)**u** + **v**.
Find the Intersection Point K The point **K** lies on both the line segment **AF** and the line segment **DE**. We can express the position vector of K in two ways:

Since K is on AF, vector AK = p * vector AF = p((1/2)**u** + **v**) for some scalar p.

Since K is on DE, vector AK = (1-q) * vector AD + q * vector AE for some scalar q. vector AK = (1-q)**v** + q(**u** + (1/4)**v**) = q**u** + (1 - 3q/4)v

By equating the coefficients of the **u** and **v** vectors, we get a system of equations:

- $p/2 = q$
- $p = 1 - 3q/4$ Solving this system gives **p = 8/11**.

Calculate the Length of AK The value p = 8/11 represents the ratio AK / AF. This means AK = (8/11)AF. The remaining segment, KF, must be the rest of the line:

KF = AF - AK = AF - (8/11)AF = (3/11)AF We are given that KF = 6.

(3/11)AF = 6

AF = $(6 \times 11) / 3 = 22$ Now we can find the length of AK:

AK = $(8/11) \times AF = (8/11) \times 22 = 16$

Problem 9:

The product of all possible values of $a^3 + b^3$ is **24**.

Create a Single-Variable Equation From the first equation, $ab^2 = 1$, we can express **a** in terms of **b**: $a = 1/b^2$. Now, substitute this into the second equation, $a^3 + 3b^3 = 4$:

$(1/b^2)^3 + 3b^3 = 4$

$1/b^6 + 3b^3 = 4$ To simplify, let $v = b^3$. The equation becomes $1/v^2 + 3v = 4$. Multiplying everything by v^2 gives:

$1 + 3v^3 = 4v^2$, which rearranges to a cubic polynomial $P(v) = 3v^3 - 4v^2 + 1 = 0$.

Relate the Expression to the New Variable The expression we want to find is $a^3 + b^3$. We can express this in terms of v as well.

Since $a = 1/b^2$, then $a^3 = 1/b^6 = 1/(b^3)^2 = 1/v^2$.

So, $a^3 + b^3 = 1/v^2 + v$. We can find a simpler form for this from our polynomial. From $3v^3 - 4v^2 + 1 = 0$, we can divide by v^2 (since v cannot be 0) to get $3v - 4 + 1/v^2 = 0$. Rearranging this gives $1/v^2 = 4 - 3v$.

Now substitute this back into our expression: $a^3 + b^3 = (4 - 3v) + v = 4 - 2v$.

Find the Product of Possible Values The cubic equation for v has three roots, let's call them v_1, v_2, v_3. This means there are three possible values for our expression $a^3 + b^3$:

$E_1 = 4 - 2v_1$

$E_2 = 4 - 2v_2$

$E_3 = 4 - 2v_3$ The question asks for the product of these values: $E_1 \times E_2 \times E_3$.

Product $= (4 - 2v_1)(4 - 2v_2)(4 - 2v_3) = 8(2 - v_1)(2 - v_2)(2 - v_3)$ The expression $(2 - v_1)(2 - v_2)(2 - v_3)$ is related to the polynomial P(v). Since $P(v) = 3(v - v_1)(v - v_2)(v - v_3)$, we know that $(2 - v_1)(2 - v_2)(2 - v_3) = P(2) / 3$.

Let's calculate P(2): $P(2) = 3(2)^3 - 4(2)^2 + 1 = 3(8) - 4(4) + 1 = 24 - 16 + 1 = 9$.

So, $(2 - v_1)(2 - v_2)(2 - v_3) = 9 / 3 = 3$.

The final product is $8 \times 3 = 24$. 🧠

Problem 10:

There are **7** EZ numbers less than 10^5.

An **EZ number** is a positive integer with at least two identical digits and exactly 4 positive divisors. A number with 4 divisors must be in one of two forms: p^3 (a prime cubed) or $p_1 \times p_2$ (the product of two distinct primes). Let's check the numbers with 2, 3, 4, and 5 digits.

Two-Digit Numbers (form dd) These numbers are $d \times 11$. Since 11 is prime, for the number to have 4 divisors, **d** must be a different prime.

d=2: $2 \times 11 = 22$ (divisors: 1, 2, 11, 22) -> **Valid**

d=3: $3 \times 11 = 33$ (divisors: 1, 3, 11, 33) -> **Valid**

d=5: $5 \times 11 = 55$ (divisors: 1, 5, 11, 55) -> **Valid**

d=7: $7 \times 11 = 77$ (divisors: 1, 7, 11, 77) -> **Valid** *(Other digits like $d=4=2^2$ or $d=6=2\times3$ would give the number more than 4 divisors).* There are **4** two-digit EZ numbers.

Three-Digit Numbers (form ddd) These numbers are $d \times 111$. The prime factorization of 111 is 3×37.

The number is d × 3 × 37. For this to have only 4 divisors, **d** must be **1**. The number is **111**. There is **1** three-digit EZ number.

Four-Digit Numbers (form dddd) These numbers are d × 1111. The prime factorization of 1111 is 11 × 101.

The number is d × 11 × 101. For this to have only 4 divisors, **d** must be **1**. The number is **1111**. There is **1** four-digit EZ number.

Five-Digit Numbers (form ddddd) These numbers are d × 11111. The prime factorization of 11111 is 41 × 271.

The number is d × 41 × 271. For this to have only 4 divisors, **d** must be **1**. The number is **11111**. There is **1** five-digit EZ number.

Total Count Adding the counts from each case:

4 (2-digit) + 1 (3-digit) + 1 (4-digit) + 1 (5-digit) = 7 🔢

Problem 11:

The correct expression is **(a + 2b) / 3**.

Find the Distance The total distance between the two numbers a and b on a number line is **b - a**.

Calculate Two-Thirds of the Distance We need to find the length that is two-thirds of this total distance.

$$(2/3) \times (b - a)$$

Find the Final Point To find the number that represents this point, we start at the smaller number, **a**, and add the distance we just calculated.

$$\text{Point} = a + (2/3)(b - a)$$

$$\text{Point} = a + (2/3)b - (2/3)a$$

$$\text{Point} = (a - (2/3)a) + (2/3)b$$

$$\text{Point} = (1/3)a + (2/3)b$$

$$\text{Point} = (a + 2b) / 3$$ ↔️

Problem 12:

The sum of the measures of ∠B and ∠C is **250°**.

Find the Sum of Angles in a Pentagon The sum of the interior angles of a polygon with **n** sides is given by the formula (n - 2) × 180°. For a pentagon, n = 5.

Sum of angles = $(5 - 2) \times 180° = 3 \times 180° = 540°$

Set Up the Equation The sum of all five angles in the pentagon is 540°. $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E = 540°$ We are given that $m\angle A = 40°$, $m\angle B = m\angle E$, and $m\angle C = m\angle D$. We can substitute these into the equation: $40° + m\angle B + m\angle C + m\angle C + m\angle B = 540°$

Solve for the Sum Now, we can combine the like terms and solve for the sum $m\angle B + m\angle C$.

$$40° + 2(m\angle B) + 2(m\angle C) = 540°$$
$$2(m\angle B + m\angle C) = 540° - 40°$$
$$2(m\angle B + m\angle C) = 500°$$
$$m\angle B + m\angle C = 500° / 2 = 250°$$ 🧠

Problem 13:

The value of $x_1 + x_n$ is **25**.

Understand the Equation The equation $2^{x1} + 2^{x2} + \ldots + 2^{xn} = 160000$ is another way of writing the number **160000** in **binary (base-2)**. The numbers $x_1, x_2, \ldots x_n$ are simply the positions of the '1's in the binary representation of 160000. Our goal is to find the smallest and largest of these positions.

Find the Binary Representation of 160000 Instead of converting the entire number, we can use its prime factorization to find the sum of powers of 2.

$160000 = 16 \times 10000 = 2^4 \times (10)^4 = 2^4 \times (2 \times 5)^4 = 2^4 \times 2^4 \times 5^4 = 2^8 \times 5^4$

Now, let's write the 5^4 part as a sum of powers of 2: $5^4 = 625 = 512 + 64 + 32 + 16 + 1$ $5^4 = 2^9 + 2^6 + 2^5 + 2^4 + 2^0$ Substitute this back into our expression for 160000:

$$160000 = 2^8 \times (2^9 + 2^6 + 2^5 + 2^4 + 2^0)$$
$$160000 = 2^{8+9} + 2^{8+6} + 2^{8+5} + 2^{8+4} + 2^{8+0}$$
$$160000 = 2^{17} + 2^{14} + 2^{13} + 2^{12} + 2^8$$

Identify x_1 and x_n The exponents in the sum are {17, 14, 13, 12, 8}. Since the problem states $x_1 < x_2 < \ldots < x_n$, we order them:

The smallest exponent is $\mathbf{x_1 = 8}$.

The largest exponent is $\mathbf{x_n = 17}$.

Calculate the Final Sum

$x_1 + x_n = 8 + 17 = 25$ 🧠

Problem 14:

The value of AD + BC is **8.**

To solve this geometry problem, we need to use two key theorems related to angle bisectors in a triangle.

Set Up Equations Using Geometric Theorems Let's call the unknown lengths AD = x and BC = y.

Angle Bisector Theorem: This theorem states that an angle bisector divides the opposite side into two segments that are proportional to the other two sides of the triangle. AD / DC = AB / BC x / (3/2) = 8 / y → xy = 12

Angle Bisector Length Formula: The square of the length of the angle bisector is equal to the product of the adjacent sides minus the product of the segments of the opposite side. $BD^2 = AB \times BC - AD \times DC$ $(3\sqrt{5})^2 = 8y - x(3/2)$ $45 = 8y - 3x/2 \rightarrow 90 = 16y - 3x$

Solve the System of Equations Now we have a system of two equations with two variables:

$$xy = 12$$
$$16y - 3x = 90$$

From the first equation, we can write x = 12/y. Substitute this into the second equation:

$$16y - 3(12/y) = 90$$
$$16y - 36/y = 90$$
$$16y^2 - 36 = 90y$$
$$16y^2 - 90y - 36 = 0$$
$$8y^2 - 45y - 18 = 0 \text{ (dividing by 2)}$$

Solving this quadratic equation gives two possible values for y, but only one is positive: **y = 6.**

Find the Lengths and the Final Sum

Since y = BC = 6, we can find x from the first equation: x * 6 = 12 → x = 2. So, **AD = 2.**

The question asks for the value of AD + BC. 2 + 6 = 8 📐

Problem 15:

There are **2** distinct values.

Simplify the GCD Expression We can use a property of the greatest common divisor (GCD) that states gcd(a, b) = gcd(a, b - k*a) for any integer k. Our goal is to use this property to eliminate the variable **n**.

Let g = gcd(6n + 15, 10n + 21).

Since g must divide both expressions, it must also divide any combination of them. Let's try to cancel the n term. The least common multiple of 6 and 10 is 30.

g must divide 5 × (6n + 15), which is 30n + 75.

g must divide 3 × (10n + 21), which is 30n + 63.

Therefore, g must also divide their difference: (30n + 75) - (30n + 63) = 12. This tells us that the GCD must be a divisor of 12. The possible values are {1, 2, 3, 4, 6, 12}.

Apply an Additional Constraint Let's look at one of the original expressions: 6n + 15.

We can factor this as 3(2n + 5).

Since 2n is always even, 2n + 5 is always odd. The product 3 × (odd number) is always **odd**.

The GCD of an odd number and any other number must be odd. Therefore, the GCD can only be an **odd divisor of 12**.

This narrows down the possibilities to just **{1, 3}**.

Check if the Values are Possible Now we just need to check if we can find values of n that produce these GCDs.

Can the GCD be 1? Let n = 1. gcd(6(1)+15, 10(1)+21) = gcd(21, 31) = 1. Yes.

Can the GCD be 3? Let n = 3. gcd(6(3)+15, 10(3)+21) = gcd(33, 51) = 3. Yes.

Since both 1 and 3 are possible values for the GCD, there are **2** distinct values.