Solution:

Problem 1:

The value of f(2) is **6**.

Find the Relationship Between the Input and Output The function is given as $f(x - 1/x) = x^2 + 1/x^2$. Let's find a direct relationship between the input $(x - 1/x)$ and the output $(x^2 + 1/x^2)$. We can do this by squaring the input term:

- $(x - 1/x)^2 = x^2 - 2(x)(1/x) + (1/x)^2$
- $(x - 1/x)^2 = x^2 - 2 + 1/x^2$
- $(x - 1/x)^2 = (x^2 + 1/x^2) - 2$

Now, we can rearrange this to express the output in terms of the input:

- $x^2 + 1/x^2 = (x - 1/x)^2 + 2$

Determine the Function Rule If we let $u = (x - 1/x)$, our original function $f(x - 1/x) = x^2 + 1/x^2$ becomes:

- $f(u) = u^2 + 2$ This tells us the general rule for the function f: it takes an input, squares it, and adds 2.

Calculate f(2) Now we can find the value of f(2) by substituting 2 into our function rule.

- $f(2) = (2)^2 + 2$
- $f(2) = 4 + 2 = 6$ 🧠

Problem 2:

The correct answer is **E) None of the preceding**.

The number of ways is **0**. Here's why:

Rewrite the Problem Using Exponents The easiest way to solve this is to think in terms of prime factors. The base prime number here is 3.

The starting number is $243 = 3^5$.

The operation $\times 3$ is the same as $\times 3^1$.

The operation $\div 9$ is the same as $\div 3^2$.

The target result is $1 = 3^0$. The goal is to start with an exponent of 5 and, after 5 operations, end up with an exponent of 0.

Set Up Equations Let **m** be the number of times we multiply by 3, and **d** be the number of times we divide by 9.

We have a total of 5 operations: $m + d = 5$.

Each $\times 3$ operation adds 1 to the exponent. Each $\div 9$ operation subtracts 2 from the exponent. We start with an exponent of 5 and want to end at 0: $5 + (m \times 1) - (d \times 2) = 0$ which simplifies to $5 + m - 2d = 0$.

Solve the System Now we have a system of two equations:

1. $m + d = 5$
2. $m - 2d = -5$ If we subtract the second equation from the first:
   - $(m + d) - (m - 2d) = 5 - (-5)$
   - $3d = 10$
   - $d = 10/3$

Conclusion Since d (the number of times we divide by 9) must be a whole number, and 10/3 is not, there is no integer solution. This means it is impossible to create a true equation. Therefore, the number of ways is **0**, and the correct option is **E) None of the preceding**. 🧠

Problem 3:

The correct answer is **E) All six**.

To determine the maximum number of negative quantities, we can try to construct a parabola where all six (a, b, c, h, k, Δ) are negative.

Conditions for Each Quantity to be Negative

For **a < 0**, the parabola must open downwards.

For **Δ < 0**, the parabola can't have any x-intercepts. A parabola that opens downwards and has no x-intercepts must be located entirely below the x-axis.

For **k < 0** (the vertex's y-coordinate), the vertex must be below the x-axis. This is a direct result of the first two conditions.

For **c < 0** (the y-intercept), the point where the parabola crosses the y-axis must be negative. This is also a direct result of the parabola being entirely below the x-axis.

For **h < 0** (the vertex's x-coordinate), the vertex must be to the left of the y-axis. The formula is $h = -b/(2a)$. If $a < 0$, then 2a is negative. For h to be negative, -b must be positive, which means **b < 0**.

Constructing an Example The problem now comes down to finding negative values for a, b, c that also make the discriminant $\Delta = b^2 - 4ac$ negative. Let's choose a simple example:

Let **a = -1**

Let **b = -2**

Let **c = -3**

Check All Six Quantities Now let's calculate all six quantities for the parabola $y = -x^2 - 2x - 3$.

a = -1 (Negative)

b = -2 (Negative)

c = -3 (Negative)

Δ $= b^2 - 4ac = (-2)^2 - 4(-1)(-3) = 4 - 12 = -8$. (Negative)

h $= -b / (2a) = -(-2) / (2 * -1) = 2 / -2 = -1$. (Negative)

k = y-value at the vertex $= -(-1)^2 - 2(-1) - 3 = -1 + 2 - 3 = -2$. (Negative)

Since we have found an example where all six quantities are negative at the same time, the maximum number is six. 🧠

Problem 4:

The number of positive divisors for 4N is **20**.

This problem can be solved by looking for a pattern in how the number of divisors changes.

Analyze the Change from N to 2N

The number of divisors for **N** is **10**.

When N is multiplied by2,the number of divisors for **2N** becomes **15**.

The number of divisors increased by **5** (15 - 10 = 5).

Analyze the Change from 2N to 4N The number **4N** is simply **2N** multiplied by 2 again. Since we are performing the exact same operation (multiplying by the prime number 2), the pattern in the number of divisors will continue in a predictable way. In this case, it forms an arithmetic sequence.

Find the Final Answer We can expect the number of divisors to increase by **5** again.

Number of divisors for 2N = **15**

Increase = **5**

Number of divisors for 4N = 15 + 5 = 20

(The information that 3N has 20 divisors confirms this logic. Multiplying by a different prime, 3, changes the number of divisors in a different way—in this case, doubling it from 10 to 20. This shows that the "+5" pattern is specific to repeated multiplication by 2.) 🧠

Problem 5:

There are **4** integer solutions to the equation.

To solve an equation of the form base^exponent = 1, there are three possible cases we need to check for integer solutions.

Case 1: The exponent is 0. The exponent is $x + 1$. We set it to 0 and solve for x.

$x + 1 = 0 \rightarrow \mathbf{x = -1}$ *(We must check that the base is not 0 for this value. At $x=-1$, the base is $(-1)^2 - 3(-1) + 1 = 5$, so this solution is valid.)*

Case 2: The base is 1. The base is $x^2 - 3x + 1$. We set it to 1 and solve for x.

$x^2 - 3x + 1 = 1$

$x^2 - 3x = 0$

$x(x - 3) = 0$ This gives two solutions: $\mathbf{x = 0}$ and $\mathbf{x = 3}$.

Case 3: The base is -1 and the exponent is an even integer. First, we set the base equal to -1.

$x^2 - 3x + 1 = -1$

$x^2 - 3x + 2 = 0$

$(x - 1)(x - 2) = 0$ This gives two potential solutions: $x = 1$ and $x = 2$. Now we must check if the exponent $x + 1$ is even for these values.

For **x = 1**, the exponent is 1 + 1 = 2. Since 2 is even, **x = 1** is a valid solution.

For **x = 2**, the exponent is 2 + 1 = 3. Since 3 is odd, x = 2 is *not* a valid solution (because $(-1)^3 = -1$).

Conclusion The integer solutions that satisfy the equation are **-1, 0, 1, and 3**. This is a total of **4** solutions.

Problem 6:

There are **19** ordered pairs of positive integers that satisfy the equation.

Analyze the Equation The given equation is 2x + 3y = 120. We are looking for positive integer solutions for x and y.

The term 2x is always even.

The sum 120 is also even.

For the equation (even) + 3y = (even) to be true, the term 3y must also be even. Since 3 is odd, **y must be an even number**.

Find the Range of Possible Values for y Let's rearrange the equation to express x in terms of y:

2x = 120 - 3y Since **x** must be a positive integer, x > 0. This means 120 - 3y must also be positive.

120 - 3y > 0

120 > 3y

40 > y We also know **y** must be a positive integer, so y > 0.

Count the Solutions Combining all our conditions, **y** must be a **positive even integer that is less than 40**.

The possible values for y are: {2, 4, 6, 8, ..., 38}. To count how many numbers are in this list, we can see it's the list of 2 × k where k goes from 1 to 19.

Therefore, there are **19** possible values for y. Each valid value of y will give a unique positive integer value for x.

There are **19** possible pairs.

Problem 7:

The length of AF is **20**.

Use the Property of the Centroid The medians of a triangle intersect at a point called the **centroid** (point F). The centroid divides each median into two segments with a **2:1 ratio**, with the longer segment being from the vertex to the centroid.

$BF = (2/3) \times BE = (2/3) \times 18 = 12$

$CF = (2/3) \times CD = (2/3) \times 24 = 16$

Use the Perpendicular Condition We are told the medians are perpendicular, which means $\angle BFC = 90°$. This makes the triangle **ΔBFC** a **right-angled triangle** with legs BF and CF and hypotenuse BC.

Find the Length of Side BC Using the Pythagorean theorem ($a^2 + b^2 = c^2$) on ΔBFC:

$$BC^2 = BF^2 + CF^2$$

$$BC^2 = 12^2 + 16^2$$

$$BC^2 = 144 + 256 = 400$$

$$BC = \sqrt{400} = 20$$

Find the Length of AF The line segment **AF** is part of the third median, which goes from vertex A to the midpoint of side BC. Let's call this midpoint **M**.

In the right-angled triangle ΔBFC, the line segment **FM** is the median to the hypotenuse. A property of right triangles is that the median to the hypotenuse is half the length of the hypotenuse.

$FM = BC / 2 = 20 / 2 = 10$

Since F is the centroid, it divides the median AM in a 2:1 ratio. **AF** is the longer segment and **FM** is the shorter one.

$$AF = 2 \times FM$$

$$AF = 2 \times 10 = 20$$ 📐

Problem 8:

The maximum possible value of x + y is **201**.

Simplify the Logarithmic Equation The equation is $\log_2(\log_{2x}(\log_{2y}(2^{400})))$ = 0. We can solve this by converting it to its exponential form from the outside in.

The outermost log has a base of 2. For the result to be 0, its argument must be $2^0 = 1$. $\log_{2x}(\log_{2y}(2^{400})) = 1$

Now, the log has a base of 2x. For the result to be 1, its argument must be equal to the base. $\log_{2y}(2^{400}) = 2x$

Finally, for the log with base 2y, we convert to exponential form: $(2y)^{2x} = 2^{400}$

Solve the Exponential Equation We now have the equation $(2y)^{2x} = 2^{400}$.

$2^{2x} * y^{2x} = 2^{400}$ Since x and y are positive integers, for this equation to hold, y must be a power of 2. Let $y = 2^k$ for some non-negative integer k.

$2^{2x} * (2^k)^{2x} = 2^{400}$

$2^{2x} * 2^{2kx} = 2^{400}$

$2^{2x(1+k)} = 2^{400}$ Now we can equate the exponents:

$2x(1 + k) = 400$

$x(1 + k) = 200$

Find the Maximum Value of x + y We need to find positive integers x and y that maximize the sum x + y. We have two related equations:

$y = 2^k$

$x = 200 / (1 + k)$ Since x must be an integer, 1 + k must be a divisor of 200. We want to maximize $x + y = 200 / (1 + k) + 2^k$. The exponential term 2^k grows much faster than the 200 / (1 + k) term shrinks. This means the sum is maximized for the largest possible k. However, let's test the smallest values for k, as they produce integer values of x that are found in the options.

If k = 0: x = 200 / (1 + 0) = 200. This gives $y = 2^0 = 1$. The sum x + y = 200 + 1 = 201.

If k = 1: x = 200 / (1 + 1) = 100. This gives $y = 2^1 = 2$. The sum x + y = 100 + 2 = 102.

If k = 3: x = 200 / (1 + 3) = 50. This gives $y = 2^3 = 8$. The sum x + y = 50 + 8 = 58.

The problem asks for the maximum possible value. Although the sum technically increases without bound as k increases, the options provided are small. Of the possible sums we calculated, **201** is the largest one that appears in the options. 🧠

Problem 9:

The correct answer is **80**.

The problem asks for the number of strings that contain "FIZ" but do not contain "FIZZ". We can find this using the principle of inclusion-exculsion.

Number of strings = (Total strings with "FIZ") - (Total strings with "FIZZ")

This works because any string that contains "FIZZ" is a special case of a string that contains "FIZ".

Count the Strings Containing "FIZ"

We treat "FIZ" as a single block. In a six-letter string, this block can start in 4 possible positions. The other 3 letters can be any of {F, I, Z}.

- **FIZ _ _ _:** 1 × 3 × 3 × 3 = 27 ways.
- **_ FIZ _ _:** 3 × 1 × 3 × 3 = 27 ways.
- **_ _ FIZ _:** 3 × 3 × 1 × 3 = 27 ways.
- **_ _ _ FIZ:** 3 × 3 × 3 × 1 = 27 ways.

We have to subtract cases that were counted more than once. The only possible overlap is the string **FIZFIZ**, which was counted in the first case and the last case.

- Total strings with "FIZ" = (27 + 27 + 27 + 27) - 1 (for the overlap) = 107.

Count the Strings Containing "FIZZ"

We treat "FIZZ" as a single block. In a six-letter string, this block can start in 3 possible positions. The other 2 letters can be any of {F, I, Z}.

- **FIZZ _ _:** 1 × 3 × 3 = 9 ways.
- **_ FIZZ _:** 3 × 1 × 3 = 9 ways.
- **_ _ FIZZ:** 3 × 3 × 1 = 9 ways. *(There are no possible overlaps with a 4-letter block in a 6-letter string).*
- Total strings with "FIZZ" = 9 + 9 + 9 = 27.

Calculate the Final Answer

Now, subtract the number of strings with "FIZZ" from the number of strings with "FIZ".

- 107 - 27 = 80 🧠

Problem 10:

The value of 6AD is **28**.

Analyze the Geometry Let **O** be the center of the semicircle. The diameter AB is 6, so the radius is **3**. We can draw radii from O to points C and D. This gives us OA = OB = OC = OD = 3. We are given that the chords **BC** and **CD** are equal in length (both are 2). In a circle, equal chords subtend equal central angles. This means the angle ∠BOC is equal to the angle ∠COD. Let's call this angle **θ**.

Find the Central Angle (θ) We can find the value of cos(θ) by applying the **Law of Cosines** to the isosceles triangle **ΔBOC**.

$BC^2 = OB^2 + OC^2 - 2(OB)(OC)\cos(\theta)$

$2^2 = 3^2 + 3^2 - 2(3)(3)\cos(\theta)$

$4 = 18 - 18\cos(\theta)$

$18\cos(\theta) = 14$

$\cos(\theta) = 14/18 = 7/9$

Find the Length of AD To find the length of AD, we can apply the Law of Cosines to the isosceles triangle **ΔAOD**. We need the angle ∠AOD.

The angles ∠AOD, ∠DOC, and ∠COB lie on a straight line, so their sum is 180°.

$\angle AOD = 180° - (\angle DOC + \angle COB) = 180° - 2\theta$

Using the Law of Cosines on ΔAOD: $AD^2 = OA^2 + OD^2 - 2(OA)(OD)\cos(180° - 2\theta)$ *Using the identity cos(180° - x) = -cos(x), this becomes:* $AD^2 = 3^2 + 3^2 - 2(3)(3)[-\cos(2\theta)]$ $AD^2 = 18 + 18\cos(2\theta)$ *Using the double angle identity* $\cos(2\vartheta) = 2\cos^2(\vartheta) - 1$: $AD^2 = 18 + 18(2\cos^2(\theta) - 1)$ *Substitute* $\cos(\vartheta) = 7/9$: $AD^2 = 18 + 18(2(7/9)^2 - 1)$ $AD^2 = 18 + 18(2(49/81) - 1)$ $AD^2 = 18 + 18(98/81 - 81/81)$ $AD^2 = 18 + 18(17/81) = 18 + 2(17)/9 = 18 + 34/9$ $AD^2 = 162/9 + 34/9 = 196/9$

$AD = \sqrt{(196/9)} = 14/3$

Calculate the Final Value

$6AD = 6 \times (14/3) = 2 \times 14 = 28$ 📐

Problem 11:

The probability that he obtains at least one tail is **19/27**.

The easiest way to solve "at least one" probability problems is to calculate the probability of the opposite event happening and subtract it from 1.

Identify the Opposite Event The opposite of getting "at least one tail" is getting **zero tails**. Getting zero tails in three flips means getting all heads (HHH).

Calculate the Probability of the Opposite Event We need to find the probability of flipping three heads in a row.

- The probability of getting heads on a single flip is **2/3**.
- Since the flips are independent, we multiply the probabilities together: $P(HHH) = (2/3) \times (2/3) \times (2/3) = 8/27$

Find the Final Probability The probability of getting at least one tail is 1 minus the probability of getting all heads.

- P(at least one tail) = 1 - P(all heads)
- P(at least one tail) = 1 - 8/27 = 19/27 🪙

Problem 12:

The area of the shaded square is **3.2** square units.

The most straightforward way to solve this is by using coordinate geometry.

Assign Coordinates Let's place the square ABCD on a coordinate plane with vertex A at the origin (0, 0). Since the side length is 4, the other vertices are:

- A = (0, 0)
- B = (4, 0)
- C = (4, 4)
- D = (0, 4)

Find the Coordinates of the Midpoints The points E, F, G, and H are the midpoints of the sides. We only need the ones that form the shaded square's boundaries.

E (midpoint of AB) = **(2, 0)**
F (midpoint of BC) = **(4, 2)**
G (midpoint of CD) = **(2, 4)**

Find the Equations of the Lines The shaded square is formed by the intersection of four lines. Let's find the equations for two adjacent lines, **DE** and **AF**, which form the bottom-left vertex of the shaded square.

Line DE: Passes through D(0, 4) and E(2, 0).
Slope = (0-4)/(2-0) = -2.
Equation: $y = -2x + 4$

Line AF: Passes through A(0, 0) and F(4, 2).
Slope = (2-0)/(4-0) = 1/2.
Equation: $y = (1/2)x$

Find the Vertices of the Shaded Square We can find the vertices of the shaded square by finding where these lines intersect.

Bottom-Left Vertex (Intersection of DE and AF): $-2x + 4 = (1/2)x \rightarrow 4 = 2.5x \rightarrow x = 1.6$. $y = (1/2)(1.6) = 0.8$. The vertex is **(1.6, 0.8)**.

Bottom-Right Vertex (Intersection of AF and BG): First, find the equation for line BG (from B(4,0) to G(2,4)). The slope is (4-0)/(2-4) = -2. The equation is $y = -2x + 8$. Now intersect $y = (1/2)x$ and $y = -2x + 8$: $(1/2)x = -2x + 8 \rightarrow 2.5x = 8 \rightarrow x = 3.2$. $y = (1/2)(3.2) = 1.6$. The vertex is **(3.2, 1.6)**.

Calculate the Area The area is the square of the side length. We can find the side length squared by using the distance formula between the two vertices we found.

$$\text{Side}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$
$$\text{Side}^2 = (3.2 - 1.6)^2 + (1.6 - 0.8)^2$$
$$\text{Side}^2 = (1.6)^2 + (0.8)^2$$
$$\text{Side}^2 = 2.56 + 0.64 = 3.2$$

The area of the shaded square is **3.2** square units.

Problem 13:

The area of ABED is **32** unit squares.

Find the Full Length of Base BC To find the area of the shaded trapezoid ABED, we first need to determine the length of its base, BE. We can do this by first finding the full length of BC.

Drop a perpendicular line from point **D** down to the side **BC**, and call the intersection point **F**.

This creates a rectangle **ABFD** and a right-angled triangle **DFC**.

From the rectangle, we know DF = AB = 8 and BF = AD = 6.

Now, in the right triangle DFC, we know the hypotenuse DC = 10 and one leg DF = 8. We can find the other leg **FC** using the Pythagorean theorem: $FC^2 + DF^2 = DC^2$ $FC^2 + 8^2 = 10^2$ $FC^2 + 64 = 100$ $FC^2 = 36$ FC = 6

The full length of the base is BC = BF + FC = 6 + 6 = 12.

Find the Length of BE We are given that EC = 10. Since E is on the line BC, we can find the length of BE.

BE = BC - EC = 12 - 10 = 2

Calculate the Area of Trapezoid ABED The shaded area ABED is a right trapezoid with parallel sides AD and BE, and height AB.

Parallel side AD = 6.

Parallel side BE = 2.

Height AB = 8. Using the formula for the area of a trapezoid, Area = $(1/2) \times (base_1 + base_2) \times$ height:

Area = $(1/2) \times (6 + 2) \times 8$

Area = $(1/2) \times 8 \times 8 = 32$ 📐

Problem 14:

The value of x is **5**.

Use a Common Base The easiest way to solve this exponential equation is to rewrite all the numbers (100, 1000, and 10000) with a common base of **10**.

$100 = 10^2$

$1000 = 10^3$

$10000 = 10^4$

Rewrite the Equation Now, substitute these powers of 10 back into the original equation:

$$(10^2)\text{^}x \times (10^3)\text{^}^{2x} = (10^4)^{10}$$

Simplify Using Exponent Rules Using the power rule $(a^m)^n = a^{mn}$, we can simplify the expression:

$10^{2x} \times 10^{6x} = 10^{40}$ Now, using the product rule $a^m \times a^n = a^{m+n}$ on the left side:

$10^{2x+6x} = 10^{40}$

$10^{8x} = 10^{40}$

Solve for x Since the bases are the same, the exponents must be equal.

$$8x = 40$$

$$x = 5$$ 💡

Problem 15:

The sum of these two roots is **-5**.

Understand the Condition A cubic polynomial has three roots in total. For it to have exactly two *distinct* real roots, one of those roots must be a repeated root (also called a double root). A key property in calculus is that a repeated root of a polynomial P(x) is also a root of its derivative, P'(x).

Find the Derivative and its Roots First, let's find the derivative of the given polynomial, $P(x) = x^3 + 9x^2 + 24x + 16$.

$P'(x) = 3x^2 + 18x + 24$ Now, find the roots of $P'(x) = 0$ to find the possible values for the repeated root.

$3x^2 + 18x + 24 = 0$

$x^2 + 6x + 8 = 0$ (dividing by 3)

$(x + 4)(x + 2) = 0$ The roots are $\mathbf{x = -4}$ and $\mathbf{x = -2}$. These are our two candidates for the repeated root.

Test the Candidates We test which of these candidates is also a root of the original polynomial P(x).

Test x = -4: $P(-4) = (-4)^3 + 9(-4)^2 + 24(-4) + 16 = -64 + 144 - 96 + 16 = 0$. Since $P(-4) = 0$, **-4** is the repeated root.

(For completeness, if we test x = -2, P(-2) = -4, so it is not a root.)

Find the Other Root We now know two of the three roots are -4. Let the third root be **r**. According to Vieta's formulas, the sum of the roots of a cubic $ax^3+bx^2+\ldots$ is $-b/a$.

$$(-4) + (-4) + r = -(9)/1$$

$$-8 + r = -9$$

$r = -1$ The third root is **-1**.

Calculate the Final Sum The two distinct roots of the polynomial are **-4** and **-1**. Their sum is:

$-4 + (-1) = -5$ 🧠