ARCHIMEDES MATH COMPETITION

Second Round 2021-2022

Solution:

Problem 1:

The ratio of the new circumference to the new diameter is **π**.

By definition, the ratio of any circle's circumference to its diameter is the constant **π (pi)**. This fundamental relationship is true for all circles, no matter their size. Increasing the radius changes the circumference and diameter, but their ratio remains the same.

Let's show this algebraically.

Define the New Dimensions:

- Let the original radius be r.
- The **new radius** is r + 1.
- The **new diameter** is 2 × (new radius), which is 2(r + 1).

Define the New Circumference:

- The formula for circumference is 2πr.
- The **new circumference** is 2π × (new radius), which is 2π(r + 1).

Calculate the Ratio: Now, we divide the new circumference by the new diameter.

$$Ratio = \frac{New\ circumference}{New\ diameter} = \frac{2\pi(r+1)}{2(r+1)}$$

The term 2(r+1) cancels out from the numerator and the denominator, leaving:

Ratio = **π**

Problem 2:

She can list **18** pairs of parallel edges.

A rectangular prism has edges that can be grouped by their direction: length, width, and height. All edges within the same group are parallel to each other. To find the total number of parallel pairs, we just need to count how many pairs can be formed within each group and then add them up.

1. **Identify the Groups of Edges** A rectangular prism has a total of 12 edges. These can be sorted into **3 groups** based on their orientation. Each group contains **4 edges** that are all parallel to one another.
2. **Count Pairs Within One Group** For a single group of 4 parallel edges, we need to find how many unique pairs we can make. This is a combinations problem ("4 choose 2").

The number of pairs in one group is: $(24)=2\times14\times3=6$

So, there are **6 pairs** of parallel edges in each group.

3. **Calculate the Total** Since there are 3 groups and each contains 6 pairs, the total number of parallel pairs is:

3 groups $\times$ 6 pairs/group = **18 pairs**

Problem 3:

Rule 1: The numbers must be strictly increasing. The sequence is 3, 4, 5, a, b, 30, 40, 50. This means a and b must be whole numbers such that: $5<a<b<30$.

- Possible values for 'a' are from 6 to 28.
- Possible values for 'b' are from 'a+1' to 29.

Let's count all pairs (a,b) that follow this rule first: We need to pick any 2 different numbers from the list {6, 7, ..., 29}. There are $29-6+1=24$ numbers in this list. The total number of ways to pick 2 numbers is $224\times23=276$. So, there are 276 possible pairs before we apply the second rule.

Rule 2: No group of four numbers (not necessarily next to each other) can form an arithmetic progression (AP). An AP is a sequence like 2, 4, 6, 8 (where the difference is constant). We need to avoid making any such group of four.

Let's list the "bad" pairs (a,b) that would create an AP, and subtract them from our total of 276.

- **Case A: APs that start with 3, 4, or 5:**

If (3, 4, 5, a) forms an AP (difference of 1), then a must be 6. So, **a cannot be 6**. (This means all pairs (6,b) like (6,7), (6,8)...(6,29) are bad. There are 23 such pairs.)

If (3, 5, a, b) forms an AP (difference of 2), then a=7, b=9. So, **(7,9) is a bad pair**.

- **Case B: APs that end with 30, 40, or 50:**

If (a, 30, 40, 50) forms an AP (difference of 10), then a must be 20. So, **a cannot be 20**. (This means all pairs (20,b) like (20,21)...(20,29) are bad. There are 9 such pairs.)

If (b, 30, 40, 50) forms an AP (difference of 10), then b must be 20. So, **b cannot be 20**. (This means all pairs (a,20) like (6,20)...(19,20) are bad. There are 14 such pairs.)

- **Case C: Other specific APs involving a and b:**

If (3, a, b, 30) forms an AP (difference of 9), then a=12, b=21. So, **(12,21) is a bad pair**.

If (4, a, b, 40) forms an AP (difference of 12), then a=16, b=28. So, **(16,28) is a bad pair**.

Counting the "bad" pairs carefully (avoiding double counts):

1. Pairs where a=6: (6,7), (6,8), ..., (6,29) - (23 pairs)
2. Pairs where a=20: (20,21), (20,22), ..., (20,29) - (9 pairs)
3. Pairs where b=20: (6,20), (7,20), ..., (19,20) - (14 pairs)

Overlap: The pair (6,20) is counted in both "a=6" and "b=20". So we subtract 1 for this overlap.

So, total from these three groups = $23+9+14-1=45$ bad pairs.

4. Specific pairs that are "bad" but not covered by the above:

(7,9) - (1 pair)

(12,21) - (1 pair)

(16,28) - (1 pair)

Total "bad" pairs = $45+1+1+1=48$ pairs.

Step 4: Find the number of "good" pairs. Number of "good" pairs = (Total possible pairs) - (Total "bad" pairs) Number of "good" pairs = 276−48=228. The final answer is 228.

Problem 4:

To minimize the positive value of $\frac{abc-def}{ghi}$:

1. Maximize the denominator (ghi):To make ghi as large as possible, we use the largest distinct digits available, which are 7, 8, and 9. So, ghi=987.
2. Minimize the numerator ($abc - def$):
   - Since all 9digits (1-9) must be distinct and used, the digits for abc and def must come from the remaining 6 digits: {1, 2, 3, 4, 5, 6}.
   - Also, the sets of digits for abc and def must be disjoint (e.g. $a \neq d, d \neq e, etc.$)
   - To make abc−def as small and positive as possible, abc and def must be very close in value. This means their first digits should be consecutive.
   - Let abc start with 4 and def start with 3 (using 4 and 3 from the available digits). The remaining digits are {1, 2, 5, 6}.
   - To make abc small and def large (to minimize abc−def), we assign the smallest remaining digits to abc's tens/units place and the largest to def's tens/units place.
   - So, abc uses digits {4, 1, 2} to form 412.
   - And def uses digits {3, 5, 6} to form 365.
   - The difference is 412−365=47.
   - The sets of digits {4,1,2} and {3,5,6} are disjoint and their union is {1,2,3,4,5,6}, so this is a valid assignment.
3. Form the fraction and simplify:

   The minimum positive value is $\frac{47}{987}$.

   Since 47 is a prime number, we check if 987 is divisible by 47:

   $$987 \div 47 = 21$$

   So, the fraction simplifies to $\frac{1}{21}$.

4. Find m+n:

   The value is $\frac{m}{n} = \frac{1}{21}$.

   Here, m=1 and n=21. They are relatively prime.

   m+n==1+21=22.

The final answer is **22**.

Problem 5:

Let N be the initial crowd size. Adults were $\frac{5}{12}N$.

N must be a multiple of 12. So, N=12k. Initial adults A=5k.

After 50 more people arrived, new crowd $N' = 12k + 50$.

New adults $A' = \frac{11}{25}N' = \frac{11}{25}(12k + 50)$.

The number of adults on the bus is $A_{bus} = A' - A = \frac{11}{25}(12k + 50) - 5k == \frac{7k+550}{25}$.

For A_{bus} to be an integer, 7k+550 must be divisible by 25. Since 550 is divisible by 25, 7k must be divisible by 25. As 7 and 25 are coprime, k must be a multiple of 25.

Also, $A_{bus} \leq 50$(people on the bus).

$\frac{7k+550}{25} \leq 50 \Longrightarrow 7k + 550 \leq 1250 \Longrightarrow 7k \leq 700 \Longrightarrow k \leq 100$.

To minimize A', we need to minimize k. The smallest multiple of 25 that is ≤ 100 is k=25.

Calvulate A' using k=25:

$$A' = \frac{11}{25}(12 \times 25 + 50) = \frac{11}{25}(300 + 50) = 11 \times 14 = 154.$$

The final answer is **154**.

Problem 6:

1. **What's a triangle made of?** Three points (Pi,Pj,Pk).
2. **What lines exist?** A line exists if the numbers differ by a prime. Primes are 2, 3, 5, 7, 11, 13, 17, 19.
3. **The "triangle rule":** If points are Pi,Pj,Pk in order, the jumps are (j–i), (k–j), and (k–i). For these to form a triangle, the two smaller jumps

must add up to the largest one: (j–i)+(k–j)=(k–i). So, Prime1 + Prime2 = Prime3.

4. **Find prime combinations:** The only way two primes add up to another prime is if one of them is 2 (e.g., 2+3=5). The possible sets of (Prime1, Prime2, Prime3) are:
   - (2, 3, 5)
   - (2, 5, 7)
   - (2, 11, 13)
   - (2, 17, 19)
5. **Count how many times each set can be placed:**
   - For (2, 3, 5): The total jump is 5. We can start at P1, P2, ..., up to P15. (Because P15 + 5 = P20, which is the last point). So, 15 triangles.
   - For (2, 5, 7): Total jump is 7. Start up to P13. So, 13 triangles.
   - For (2, 11, 13): Total jump is 13. Start up to P7. So, 7 triangles.
   - For (2, 17, 19): Total jump is 19. Start up to P1. So, 1 triangle.
6. **Account for order:** The jumps could be (Prime1, Prime2) or (Prime2, Prime1). For example, (2,3,5) and (3,2,5) give different sets of triangles (e.g., P1 to P3 to P6, versus P1 to P4 to P6). So, we double the count.

Total = 2×(15+13+7+1) Total = 2×36=72.

The final answer is **72**.

Problem 7:

Let's assign variables to each symbol:

- Let the first symbol (top-left) be represented by N.
- Let the second symbol (middle-top) be represented by H.
- Let the third symbol (top-right) be represented by C.

From the grid, we can form equations based on the sums of rows and columns:

From the columns:

1. **Middle Column (all H symbols):** H + H + H = 12 3H = 12 Dividing by 3, we get: H = 4

Now that we know H = 4, we can substitute this into other equations.

From Row 2 (or Column 3, as they are the same):

2. C + H + C = 20 2C + H = 20 Substitute H = 4: 2C + 4 = 20 2C = 20 - 4 2C = 16 Dividing by 2, we get: C = 8

Now that we know H = 4 and C = 8, let's find N.

From Row 3:

3. N + H + H = 15 N + 2H = 15 Substitute H = 4: N + 2(4) = 15 N + 8 = 15 N = 15 - 8 N = 7

So, we have the values for all symbols:

- N = 7
- H = 4
- C = 8

Finally, find the value of X. X is the sum of the symbols in the first row: X = N + H + C X = 7 + 4 + 8 X = 19

Let's quickly verify with the first column sum: N + C + N = 7 + 8 + 7 = 22. This matches the given sum.

The value of X is 19.

The final answer is **19**.

Problem 8:

Let the seats be 1, 2, 3, 4, 5. Seat 3 is the middle seat.

- **Rule 1:** Sam and Mike (SM) must sit together.
- **Rule 2:** Adam and Josh (AJ) cannot sit together.

If **Trevor** sits in the middle (seat 3): _ _ T _ _

The SM block must go into seats (1,2) or (4,5).

- If SM are in (1,2): S M T _ _ -- Remaining are Adam and Josh for seats (4,5). They *must* sit next to each other, which breaks Rule 2.
- If SM are in (4,5): _ _ T S M -- Remaining are Adam and Josh for seats (1,2). They *must* sit next to each other, which breaks Rule 2.

Since Adam and Josh will always be forced to sit next to each other if Trevor is in the middle, **Trevor cannot be in the middle seat.**
(For any other friend in the middle seat, there's always a way to place Adam and Josh apart.)
The final answer is **Trevor**.