Second Round 2021-2022

Solution:

Problem 1:

The smallest five-digit number that meets the conditions is **12,384**.

The number must satisfy three conditions: it's a five-digit number, its digits are all different, and it's divisible by each of its own digits. Here are a few key deductions from these rules:

- The digit **0 cannot be used**, as division by zero is impossible.
- If the digit **5** were used, the number would have to end in 5. However, if any even digit (2, 4, 6, 8) were also used, the number would have to be even. A number can't be both. To find the smallest number, it's best to use the smaller digits, so we'll exclude 5.
- If the digit **3** is used, the sum of the digits must be divisible by 3.

Find the Right Set of Digits To make the number as small as possible, we should use the smallest available digits. Since we're excluding 0 and 5, we need to find a set of five digits whose sum is a multiple of 3. The set with the smallest possible sum that meets this rule is **{1, 2, 3, 4, 8}**, as their sum is 18.

Apply Divisibility Rules The number must be divisible by 1, 2, 3, 4, and 8.

Divisibility by 3: Guaranteed, as the sum is 18.

Divisibility by 8: This is the strictest rule, as it also guarantees divisibility by 2 and 4. The last three digits of the number must form a number divisible by 8.

Find the Smallest Arrangement We need to arrange the digits {1, 2, 3, 4, 8} into the smallest possible number that follows the rules.

The number must start with **1** to be as small as possible.

Testing combinations, the smallest number where the last three digits are divisible by 8 is **12,384**. Let's check it:

Divisible by 1, 2, 3? Yes.

Divisible by 4? Yes, because the last two digits, 84, are divisible by 4.

Divisible by 8? Yes, because the last three digits, 384, are divisible by 8 (384 ÷ 8 = 48).

Since this is the first valid number we find when building from the smallest digits upwards, it is the correct answer.

Problem 2:

They would represent our 100 as **144**.

The Xenon inhabitants use a number system with eight digits (0-7), which is known as a **base-8** or **octal** system. Our standard system uses ten digits (0-9) and is called **base-10**. To solve the problem, we just need to convert the base-10 number 100 into its base-8 equivalent.

The simplest way to convert from base-10 to base-8 is by using repeated division. We divide the number by 8 and record the remainders until the result is zero.

1. Divide 100 by 8: 100÷8=12 with a remainder of **4**.
2. Now, divide the result (12) by 8: 12÷8=1 with a remainder of **4**.
3. Finally, divide that result (1) by 8: 1÷8=0 with a remainder of **1**.

To get the final answer, you read the remainders from the bottom up. This gives you **144**.

Problem 3:

Mr. Smithfield can expect to pay a total of **$26.95**.

The "buy 5 get 2 free" deal means that for every **7 packs** of sausages Mr. Smithfield gets, he only needs to pay for **5** of them. To find the total cost for 15 packs, we first need to figure out how many of these 7-pack bundles he can use and how many individual packs he'll need to buy after that.

1. **Figure Out the Bundles:** Mr. Smithfield needs 15 packs. He can take advantage of the "7 for the price of 5" deal twice.
   2 bundles × 7 packs/bundle = 14 packs
2. **Determine Remaining Packs:** After getting 14 packs through the deal, he still needs one more pack.
   15 packs needed - 14 packs from bundles = 1 pack remaining
3. **Calculate Total Paid Packs:** Now, we count how many packs he actually has to pay for.
   For the first bundle, he pays for **5 packs**.
   For the second bundle, he pays for another **5 packs**.
   For the last one, he pays for **1 pack**.
   Total packs to pay for = 5 + 5 + 1 = **11 packs**.
4. **Calculate the Total Cost:** Multiply the number of paid packs by the price per pack.
   11 packs × $2.45/pack = **$26.95**

Problem 4:
The greatest multiple of 3 that can be formed is **852**.

The key to solving this is the divisibility rule for 3: a number is divisible by 3 if the **sum of its digits** is also divisible by 3. To find the greatest possible number, we should try to use as many of the given digits as possible, arranged in descending order.

1. **Check All Four Digits** First, let's see if we can use all four digits: {2, 4, 5, 8}.
   Sum = 2 + 4 + 5 + 8 = 19 Since 19 is not divisible by 3, we **cannot** make a 4-digit multiple of 3 using these digits.
2. **Check Combinations of Three Digits** Next, we'll try using three digits. We need to remove one digit from the original set so the sum of the remaining three is divisible by 3.
   The total sum is 19. If we remove the digit **4**, the new sum is

19 - 4 = 15.

Since 15 is divisible by 3, the set of digits **{2, 5, 8}** can form a multiple of 3. (Removing any of the other digits does not result in a sum divisible by 3).

3. **Form the Greatest Number** To make the greatest possible number with the digits {2, 5, 8}, we simply arrange them in descending order.

This gives us the number **852**.

Problem 5:

There are **zero** possibilities for the number.

Here's a quick way to check the possibilities:

1. A number that is **4 more than a multiple of 5** must end in a 4 or a 9. The possibilities under 50 are: 4, 9, 14, 19, 24, 29, 34, 39, 44, 49
2. Of those, the number must be **3 more than a multiple of 4**. This narrows our list down to just two numbers:
   19 ($4 \times 4 + 3$)
   39 ($4 \times 9 + 3$)
3. Finally, the number must be **5 more than a multiple of 3** (meaning it has a remainder of 2 when divided by 3).
   $19 \div 3$ has a remainder of **1**. (Doesn't work)
   $39 \div 3$ has a remainder of **0**. (Doesn't work)

Since no number satisfies all three conditions, there are no possible solutions.

Problem 6:

There are two straightforward ways to solve this:

1. The Combinations Formula

We are choosing groups of **2** points from a set of **10**. The formula for this is "10 choose 2", or $\binom{10}{2}$.

$$\binom{10}{2} = \frac{10 \times 9}{2 \times 1} = \frac{90}{2} = 45$$

2. Logical Summation

You can also think about it by starting at one point and drawing lines.

- The **1st** point can connect to the 9 other points, creating **9** chords.
- The **2nd** point can connect to the 8 *remaining* points (since its chord to the 1st point is already counted), creating **8** new chords.
- The **3rd** point can connect to the 7 remaining points, creating **7** new chords.
- ...and so on, until the 9th point connects to the last one.

The total is the sum of this pattern: 9+8+7+6+5+4+3+2+1=45

Problem 7:

The value of **A + B + C** is **97**.

While you could solve for each variable (A, B, and C) individually, there's a much faster way to solve this problem. By adding all three equations together, you can find the value of A + B + C directly.

1. **Write down the three equations:**
   - A + 2B = 92
   - 3A + C = 126
   - 2B + 3C = 170
2. **Add all three equations together:** When you add the left and right sides, you get: (A + 2B) + (3A + C) + (2B + 3C) = 92 + 126 + 170
3. **Combine the like terms:**
   - **A's:** A + 3A = 4A
   - **B's:** 2B + 2B = 4B
   - **C's:** C + 3C = 4C
   - **Numbers:** 92 + 126 + 170 = 388

The combined equation is: 4A + 4B + 4C = 388

4. **Solve for (A + B + C):** Factor out the 4 from the left side: 4(A + B + C) = 388

Now, just divide both sides by 4: A + B + C = 388 / 4

A + B + C = 97

Problem 8:

The last time this happened before 2014 was in the year **1709**.

We are looking for a year ABCD where the digits are unique and $A + B + C < D$.

- Years in the **1900s** and **1800s** are impossible. For a year like 19CD, the sum 1+9+C is at least 10, which can never be less than a single digit D. A similar issue occurs for the 1800s.
- In the **1700s**, for a year 17CD, the rule becomes $1 + 7 + C < D$, which simplifies to $\mathbf{8 + C < D}$.
- For this condition to be possible, C must be **0**, which requires D to be greater than 8. The only digit that works is **9**.
- This gives us the year **1709**. It fits both rules:
  - The digits (1, 7, 0, 9) are all different.
  - The sum $1 + 7 + 0 = 8$, which is less than the last digit, 9.