Solution:

Problem 1:

Let's assume some easy numbers:

1. Mower Speeds:

- Curt's mower is the slowest base. Let's say **Curt's mower cuts 1 unit of lawn per hour**. (SC=1)
- Bert's mower is twice as fast as Curt's: $SB=2\times SC=2\times 1=2$ units per hour.
- Andrew's mower is three times as fast as Curt's: $SA=3\times SC=3\times 1=3$ units per hour.

2. Lawn Sizes:

- Andrew's lawn is related to both Bert's and Curt's. Let's pick a size for Andrew's lawn that is easy to divide by 2 and 3. Let's say **Andrew's lawn is 6 units big**. (LA=6)
- Andrew's lawn is twice as big as Bert's: $6=2\times LB \Longrightarrow LB=3$ units.
- Andrew's lawn is three times as big as Curt's: $6=3\times LC \Longrightarrow LC=2$ units.

Now, let's calculate the time each person takes to mow their own lawn. (Time = Lawn Size / Mower Speed)

- **Andrew's Time:** Lawn Size = 6 units Mower Speed = 3 units/hour Time = $6/3=2$ hours.
- **Bert's Time:** Lawn Size = 3 units Mower Speed = 2 units/hour Time = $3/2=1.5$ hours.
- **Curt's Time:** Lawn Size = 2 units Mower Speed = 1 unit/hour Time = $2/1=2$ hours.

Comparing the times:

- Andrew: 2 hours

- Bert: 1.5 hours
- Curt: 2 hours

Bert finishes in the shortest amount of time (1.5 hours).

The final answer is **Bert**.

Problem 2:

Distances between Sphere Centers: Use the fact that mutually externally tangent spheres have the sum of their radii as the distance between their centers. This gives the lengths of the sides of the triangle formed by $O_1,\ O_2,\ O_3$.

Sphere-Plane Intersection Geometry: Understand that a plane intersecting a sphere forms a circle. The relationship between the sphere's radius (r_i), the distance from its center to the plane (d_i), and the intersection circle's radius (r_c) is given by the Pythagorean theorem: ${r_i}^2 = {d_i}^2 + {r_c}^2$. This allows calculating ${d_i}^2$ in terms of ${r_c}^2$.

3D Distance Relation: Relate the 3D distance between two sphere centers (O_iO_j) to the 2D distance between their projected centers on the plane (A_iA_j) and the difference in their perpendicular distances to the plane ($d_i - d_j$). This forms a 3D right triangle where ${O_iO_j}^2 = {A_iA_j}^2 + (d_i - d_j)^2$.

Systematic Solution of Equations: Use these relationships to create a system of equations. Solve for the unknown d_i values and ${r_c}^2$. The “same side of the plane” condition ensures d_i values are positive, helping to resolve ambiguities (like $d_i - d_j = \pm 4$).

Final Calculation: Apply the 3D distance relation once more with the calculated d_i values to find requested AC^2.

The final answer is 756.

Problem 3:

Main Ideas:

Tangent Properties: Use the tangent-secant theorem and equal tangent segments from a point to a circle (AY=AX, CZ=CX′). This helps find initial segment lengths (AX=6,CZ=20).

Parallelogram Geometry: Relate the sides of the parallelogram (AB,AD) using the calculated segments (AB=AX+BX, AD=AY+YD=BX+CZ). Note that the height between parallel sides AD and BC is 2r (circle tangent to both).

Angle Bisector Property: The angle bisectors of consecutive angles in a parallelogram (∠A and ∠B) meet at a 90∘ angle (△AOB is a right triangle). This leads to r^2=AX·BX.

Trigonometric Relation: Use the cosine of ∠A derived from △AOB ($cosA = (36 - r^2)/(36 + r^2)$).

Law of Cosines: Apply the Law of Cosines to △ADC using the diagonal AC and the expression for cosA. This allows solving for BX.

Area Calculation: Once BX is found, calculate the area of the parallelogram using Area=AD×(2r).

$$AX = 6\ (from\ AP \cdot AQ = AX^2 \Rightarrow 3 \cdot 12 = AX^2).$$
$$CZ = 20\ (from\ CQ \cdot CP = CZ^2 \Rightarrow 16 \cdot 25 = CZ^2).$$

Let $BX = k$, then $r^2 = 6k$.

$$\cos(\angle DAB) = \frac{6-k}{6+k}.$$

Parallelogram sides: AB=6+k, AD=k+20.

Apply Law of Cosines on △ADC:

$AC^2 = AD^2 + CD^2 - 2(AD)(CD)\cos(\angle ADC)$.

$28^2 = (k+20)^2 + (6+k)^2 + 2(k+20)(6+k)(\frac{6-k}{6+k})$.

…

k=4.5

Calculate area:

$$Area = AD \times 2r = (k+20) \times 2\sqrt{6k}$$

m=147, n=3.

m+n=147+3=150.

The final answer is **150**.

Problem 4:

Main Idea: To find Carl's total winning probability, calculate his chance of winning the tournament for each possible semifinal pairing and sum these probabilities. Each pairing is equally likely.

Key Probabilities:

- P(Azar beats Carl) = 2/3 ⟹ P(Carl beats Azar) = 1/3
- P(Azar/Carl beats Jon/Sergey) = 3/4 ⟹ P(Jon/Sergey beats Azar/Carl) = 1/4

Calculations:

Semifinal Pairings (each with 1/3 probability):

- **(A vs C) & (J vs S):**

Carl wins SF (vs Azar): 1/3.

Carl wins Final (vs Jon or Sergey, always 3/4 chance): 3/4.

Probability for this pairing: $(1/3)\times(1/3\times3/4)=1/12$. (No, this is wrong. It is $(1/3)\times(1/3)\times(3/4)=1/12$)

Let's re-state the probability for each specific pairing: P(Carl wins | Pairing) = P(Carl wins SF) * P(Carl wins Final)

Pairing 1: (A vs C) & (J vs S)

P(Carl wins SF vs A) = 1/3.

P(Carl wins Final vs (J or S)) = 3/4 (since $P(C>J)=3/4$ and $P(C>S)=3/4$, the specific opponent from J/S doesn't change Carl's final win probability).

Probability for Carl to win *given this pairing*: $(1/3)\times(3/4)=1/4$.

Contribution to total probability: $(1/3 \text{ for pairing})\times(1/4)=1/12$.

- **Pairing 2: (A vs J) & (C vs S)**

P(Carl wins SF vs S) = 3/4.

Carl's final opponent could be Azar (with $P(A>J)=3/4$) or Jon (with $P(J>A)=1/4$).

P(Carl wins Final) = $P(A>J)P(C>A) + P(J>A)P(C>J) = (3/4)(1/3)+(1/4)(3/4)=1/4+3/16=7/16$.

Probability for Carl to win *given this pairing*: $(3/4)\times(7/16)=21/64$.

Contribution to total probability: $(1/3 \text{ for pairing})\times(21/64)=7/64$.

- **Pairing 3: (A vs S) & (C vs J)**

This is symmetric to Pairing 2.

Contribution to total probability: 7/64.

Total Probability (p/q): Total P = 1/12+7/64+7/64=1/12+14/64=1/12+7/32.

Total $P = \frac{8}{96} + \frac{21}{96} = \frac{29}{96}$.

Final Result (p+q): p=29,q=96. (Relatively prime, as 29 is prime and 96 is not a multiple of 29). p+q=29+96=125.

The final answer is 125.

Problem 5:

The remainder is **4**.

First, the general term of the sum, $\binom{\binom{n}{2}}{2}$, simplifies algebraically to $3\binom{n+1}{4}$. This turns the problem into finding the sum $S = \sum_{n=3}^{40} 3\binom{n+1}{4}$. Using the Hockey-stick identity, this entire sum collapses to a single expression:

$S = 3 \times \binom{42}{5}$.

The value of this expression is 3×850,668=2,552,004. When 2,552,004 is divided by 1000, the remainder is 4.

Problem 6:

To maximize $x_{76} - x_{16}$ under the given sum constraint, concentrate all negative values on x_1 to x_{16} and all positive values on x_{76} to x_{100}, setting all intermediate terms to zero.

The correct answer is **841**.