ARCHIMEDES MATH COMPETITION

First Round 2021-2022

Solution:

Problem 1:

We need to find the number of positive integers less than 500 whose digits sum to 5. Any such number can have one, two, or three digits. Let's think of them all as three-digit numbers of the form abc where a,b,c are the digits. For a two-digit number, a=0. For a one-digit number, a=0 and b=0. The condition is that the sum of the digits is 5: a+b+c=5

Since the integer must be less than 500, the hundreds digit, a, can only be 0, 1, 2, 3, or 4.

Let's count the number of solutions for b and c for each possible value of a:

1. **If a=0 (Numbers below 100):** We need b+c=5. The pairs (b,c) can be (0,5), (1,4), (2,3), (3,2), (4,1), (5,0). This gives **6** numbers (which are 5, 14, 23, 32, 41, 50).
2. **If a=1 (Numbers in the 100s):** We need b+c=4. There are 4+1=5 solutions for (b,c). This gives **5** numbers.
3. **If a=2 (Numbers in the 200s):** We need b+c=3. There are 3+1=4 solutions for (b,c). This gives **4** numbers.
4. **If a=3 (Numbers in the 300s):** We need b+c=2. There are 2+1=3 solutions for (b,c). This gives **3** numbers.
5. **If a=4 (Numbers in the 400s):** We need b+c=1. There are 1+1=2 solutions for (b,c). This gives **2** numbers.

Now, we add up the counts for each case: Total numbers = 6+5+4+3+2=20.
Correct answer: **B)**

Problem 2:

Let's compare the number of students in each room type using a ratio.

- The ratio of students in **double rooms** to students in **single rooms** is 75:25, which simplifies to **3:1**.

Now, let's find the number of rooms needed for this ratio.

- For every **3** students in double rooms, you need $3 \div 2 = 1.5$ double rooms.
- For every **1** student in a single room, you need $1 \div 1 = 1$ single room.

This means the ratio of **double rooms** to the **total rooms** is 1.5 out of (1.5+1), which is 1.5 out of 2.5.

Finally, calculate the percentage:

$$\text{Percentage} = \frac{1.5}{2.5} \times 100\% = \frac{15}{25} \times 100\% = \frac{3}{5} \times 100\% = 60\%$$

Correct answer: **E)**

Problem 3:

Step 1: Identify the Numbers

First, let's list the first 7 prime numbers and the divisor.

- **First 7 prime numbers:** 2, 3, 5, 7, 11, 13, 17
- **The product (P):** $P = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$
- **The divisor:** 510

Step 2: Find the Prime Factorization of the Divisor

To see if the product is divisible by 510, we find the prime factors of 510.

- $510 = 10 \times 51$
- $10 = 2 \times 5$
- $51 = 3 \times 17$

So, the prime factorization of 510 is $2 \times 3 \times 5 \times 17$.

Step 3: Compare and Conclude

Now, compare the prime factors of 510 with the prime numbers in the product P.

- **Product P:** $P = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17$
- **Divisor 510:** $510 = 2 \times 3 \times 5 \times 17$

All the prime factors of 510 are present in the product of the first 7 prime numbers. This means the product is a multiple of 510.

When a number is divided by one of its factors, the remainder is always 0.

✅

Correct answer: **A)**

Problem 4:

There are **7** awesome numbers between 1 and 25.

An "awesome" number has exactly four factors. For a number to have exactly four factors, it must be in one of two forms:

1. The product of two different prime numbers ($p\times q$).
2. The cube of a prime number (p3).

Let's find the numbers that fit these patterns between 1 and 25.

Numbers of the form $p\times q$

We'll multiply different prime numbers together to see which products are less than 25.

- $2\times3=6$
- $2\times5=10$
- $2\times7=14$
- $2\times11=22$
- $3\times5=15$
- $3\times7=21$

Any other product of two different primes (like $2\times13=26$ or $3\times11=33$) is greater than 25. This gives us **6** numbers.

Numbers of the form p3

We'll find the cube of prime numbers to see which are less than 25.

- 23=8
- 33=27 (This is too large)

This gives us **1** number.

Total Count

By combining both lists, we get all the awesome numbers between 1 and 25: 6, 8, 10, 14, 15, 21, 22

Counting them up, we have a total of **7** numbers.

Correct answer: **B)**

Problem 5:

To solve this, we need to find the Least Common Multiple (LCM) of the three time intervals. The LCM is the smallest number that is a multiple of all three intervals. This will give us the time in seconds when they first flash together again.

Step 1: Find the LCM in Seconds

First, find the prime factorization of each number:

- $8 = 2^3$
- $9 = 3^2$
- $10 = 2 \times 5$

To find the LCM, we take the highest power of each prime factor present and multiply them together.

- Highest power of 2 is 2^3.
- Highest power of 3 is 3^2.
- Highest power of 5 is 5^1.

$$LCM = 2^3 \times 3^2 \times 5 = 8 \times 9 \times 5 = 360$$

So, the lights will flash together every **360 seconds**. 💡

Step 2: Convert Seconds to Minutes

The question asks for the answer in minutes, so we convert our result.

$$360 \text{ seconds} \div 60 \frac{\text{seconds}}{\text{minute}} = 6 \text{ minutes}$$

Correct answer: **E)**

Problem 6:

Jean can make a maximum of **12** half-pound bags of trail mix.

Here's how to get the answer.

Cost of One Pound of Trail Mix

First, we need to find the cost of a single one-pound batch of the mix. The ingredients are mixed in a ratio of 4 parts walnuts, 1 part raisins, 3 parts peanuts, and 2 parts almonds, for a total of 4+1+3+2=10 parts.

In a one-pound mix, the weight and cost of each ingredient would be:

- **Walnuts:** 0.4 lbs × $9/lb = $3.60
- **Raisins:** 0.1 lbs × $5/lb = $0.50
- **Peanuts:** 0.3 lbs × $8/lb = $2.40
- **Almonds:** 0.2 lbs × $12/lb = $2.40

The total cost for one pound of trail mix is the sum of these costs:

Cost1lb=$3.60+$0.50+$2.40+$2.40=$8.90

Cost Per Half-Pound Bag

Since each bag is a half-pound, the cost per bag is half the cost of one pound.

$$Cost_{bag} = \frac{\$8.90}{2} = \mathbf{\$4.45}$$

Maximum Number of Bags

Now we can find how many bags Jean can make with her $54.

$$Maximum\ Bags = \frac{\text{Total Money}}{\text{Cost per Bag}} = \frac{\$54}{\$4.45} \approx 12.13$$

Since she can only make whole bags, we take the integer part of the result. She can make **12 bags** and will have some money left over. (12×$4.45=$53.40)

Correct answer: **C)**

Problem 7:

Jack's team got **3** answers incorrect.

Here is the explanation for the solution.

Reasoning with a Perfect Score

Let's first figure out the maximum possible score. If the team answered all 20 questions correctly, their score would be:

20 questions×3 points/question=60 points

Their actual score was **48 points**. The difference between a perfect score and their score is:

60−48=12 points

Points Lost Per Incorrect Answer

Now, let's see what happens every time the team gets an answer wrong instead of right.

- They lose the **3 points** they would have gotten for a correct answer.
- They also have **1 point subtracted** for the incorrect penalty.

So, for each incorrect answer, the team's score drops by a total of 3+1=4 points compared to a correct answer.

Calculating the Number of Incorrect Answers

Since the team's total score was 12 points lower than a perfect score, and each incorrect answer costs 4 points, we can find the number of incorrect answers by dividing the total points lost by the points lost per incorrect answer.

Incorrect Answers=4 points per incorrect answer12 points lost=3

The team answered **3** questions incorrectly.

Correct answer: **D)**

Problem 8:

There are **4360 calories** in 7000 grams of the lemonade.

Calories in the Original Recipe

First, let's find the total calories and weight of Jasmine's original batch.

- **Weight of original batch:** 200 g (lemon juice)+100 g (sugar)+400 g (water)=700 grams.
- **Calories in original batch:**
  - From lemon juice: 200 g×0.25 cal/g=50 calories
  - From sugar: 100 g×3.86 cal/g=386 calories

- Total calories = 50+386=436 calories.

So, a **700-gram** batch of lemonade contains **436 calories**.

Calories in 7000 Grams

Now, we need to find the calories in a **7000-gram** batch. This new amount is exactly **10 times larger** than the original 700-gram recipe (7000÷700=10). To find the calories in the larger batch, we simply multiply the original calorie count by this scaling factor of 10.

$$Total\ Calories = 436\ calories \times 10 = 4360\ calories$$

Correct answer: **A)**

Problem 9:

It would take Jane **40 hours** to paint the house by herself. 🏡

Here's the logical breakdown of the problem.

Analyze the Scenarios

The problem gives us two scenarios. Let's use the second, more complex scenario first.

When the house is half-finished, Jane stops. The total job takes **30 hours**.

- **First Half:** Arno and Jane work together. We know from the first scenario that if they worked together the whole time, it would take 20 hours. Therefore, to complete the first half together, it takes them: 20 hours÷2=10 hours
- **Second Half:** Since the total time was 30 hours and the first half took 10 hours, the second half must have taken: 30 hours−10 hours=20 hours Only **Arno** worked during this time.

Calculate the Painting Rates

From the analysis above, we can figure out how fast each person paints.

- **Arno's Time:** If it takes Arno 20 hours to paint half a house, it would take him 20×2=40 hours to paint the entire house by himself.
- **Jane's Time:** We know their combined time is 20 hours. We can use their individual rates (the fraction of a house they paint per hour) to find Jane's time.
  - Arno's rate is 1/40 of the house per hour.

- Their combined rate is 1/20 of the house per hour.
- Jane's rate = (Combined Rate) - (Arno's Rate)
- Jane's rate = 1/20-1/40=2/40-1/40=1/40

Find Jane's Total Time

Since Jane's rate is painting 1/40 of the house per hour, it would take her **40 hours** to paint the entire house alone.

Correct answer: **B)**

Problem 10:

The speed of the faster runner is **5 m/s**.

To find the answer, we need to calculate the speed of each runner in meters per second (m/s) and then compare them.

Katie's Speed

First, we find the total distance Katie runs and then calculate her speed.

- **Distance:** 3 laps × 160 meters/lap = 480 meters
- **Time:** 120 seconds
- **Speed:** $\frac{480\ meters}{120\ seconds} = 4m/s$

Lottie's Speed

Next, we do the same calculation for Lottie.

- **Distance:** 5 laps × 160 meters/lap = 800 meters
- **Time:** 160 seconds
- **Speed:** $\frac{800\ meters}{160\ seconds} = 5m/s$

Conclusion

Comparing their speeds, Lottie (5 m/s) is faster than Katie (4 m/s).The question asks for the speed of the faster runner. 🏃

Correct answer: **D)**

Problem 11:

A box has 12 edges (4 lengths, 4 widths, 4 heights). The total 72-m wire means the sum of the dimensions is:

$$L + W + H = \frac{72}{4} = 18\ m$$

To get the largest possible volume, the box must be a **cube**. Therefore, each dimension must be 18÷3=6 m.

The maximum volume is:

$$V = 6 \times 6 \times 6 = 216\, m^3$$

Correct answer: **B)**

Problem 12:

The Triangle Inequality Theorem

To solve this, we use a rule called the **Triangle Inequality Theorem**. It states that the length of any side of a triangle must be less than the sum of the other two sides and greater than their difference.

To find the smallest possible value for side *S*, we only need the "difference" part of the rule:

Side1−Side2<Side3

Calculation

Let's plug in the given side lengths:

10 - 6.5 < 3.5 < S

This means that the length of side *S* must be greater than 3.5. Since the problem states that *S* is a **whole number**, the smallest whole number greater than 3.5 is **4**.

Correct answer: **E)**

Problem 13:

Correct answer: **B)**

Problem 14:

Explanation

To find the largest value, we can use the laws of exponents to rewrite the numbers so they have a common base or a common exponent, which makes them easier to compare.

Correct answer: **B)**

Problem 15:

The value of **x** is **40**.

To solve this equation, we need to simplify both sides so they have the same base. Remember that for power towers like these, we evaluate from the top down.

Correct answer: **D)**